Stats for Engineers Lecture 11
Acceptance Sampling Summary

**Acceptable quality level:** $p_1$

(consumer happy, want to accept with high probability)

**Unacceptable quality level:** $p_2$

(consumer unhappy, want to reject with high probability)

**Producer’s Risk:** reject a batch that has acceptable quality

$\alpha = P(\text{Reject batch when } p = p_1)$

**Consumer’s Risk:** accept a batch that has unacceptable quality

$\beta = P(\text{Accept batch when } p = p_2)$

**One stage plan:** can use table to find number of samples and criterion

**Two stage plan:** more complicated, but can require fewer samples

Operating characteristic curve $L(p)$: probability of accepting the batch
Is acceptance sampling a good way of quality testing?

Problems:

It is too far downstream in the production process; better if you can identify where things are going wrong.

It is 0/1 (i.e. defective/OK) - not efficient use of data; large samples are required.

- better to have quality measurements on a continuous scale: earlier warning of deteriorating quality and less need for large sample sizes.

Doesn’t use any information about distribution of defective rates
Which of the following has an exponential distribution?

1. The time until a new car’s engine develops a leak
2. The number of punctures in a car’s lifetime
3. The working lifetime of a new hard disk drive
4. 1 and 3 above
5. None of the above

Exponential distribution gives the time until or between random independent events that happen at constant rate.
(2) is a discrete distribution.
(1) and (3) are times to random events, but failure rate almost certainly increases with time.
Reliability

Problem: want to know the time till failure of parts

E.g.

- what is the mean time till failure?
- what is the probability that an item fails before a specified time?

If a product lasts for many years, how do you quickly get an idea of the failure time?

**accelerated life** testing:

**Compressed-time testing:** product is tested under usual conditions but more intensively than usual (e.g. a washing machine used almost continuously)

**Advanced-stress testing:** product is tested under harsher conditions than normal so that failure happens soon (e.g. refrigerator motor run at a higher speed than if operating within a fridge). - requires some assumptions
How do you deal with items which are still working at the end of the test programme?

An example of *censored data*.  
– we don’t know all the failure times at the end of the test

**Exponential data** (failure rate $\nu$ independent of time)

Test components up to a time $t_0$

- assuming a rate, can calculate probability of no failures in $t_0$.

- calculate probability of getting any set of failure times (and non failures by $t_0$)

- find maximum-likelihood estimator for the failure rate in terms of failure times

For failure times $t_i$, with $t_i = t_0$ for parts working at $t_0$, and $n_f$ failures

$$\Rightarrow \text{Estimate of failure rate is } \hat{\nu} = \frac{n_f}{\sum_i t_i}$$  [see notes for derivation]
Example:
50 components are tested for two weeks. 20 of them fail in this time, with an average failure time of 1.2 weeks.

What is the mean time till failure assuming a constant failure rate?

Answer:

\[ n = 50, \ n_f = 20 \]

\[ \sum_i t_i = 20 \times 1.2 + 30 \times 2 = 84 \text{ weeks} \]

\[ \Rightarrow \hat{\nu} = \frac{n_f}{\sum_i t_i} = \frac{20}{84} = 0.238/\text{week} \]

\[ \Rightarrow \text{mean time till failure is estimated to be } \frac{1}{\hat{\nu}} = \frac{1}{0.238} = 4.2 \text{ weeks} \]
Reliability function and failure rate

For a pdf $f(x)$ for the time till failure, define:

**Reliability function**

Probability of surviving at least till age $t$. i.e. that failure time is later than $t$

$$R(t) = P(T > t) = \int_t^\infty f(x)dx$$

$$= 1 - F(t)$$

$F(t) = \int_0^t f(t)dt$ is the cumulative distribution function.

**Failure rate**

This is failure rate at time $t$ given that it survived until time $t$:

$$\phi(t) = \frac{f(t)}{R(t)}$$

$$P(\text{fail at } t | \text{OK until } t) = \frac{P(\text{OK until } t \cap \text{fail at } t)}{P(\text{OK until } t)} = \frac{f(t)}{R(t)}$$
Example: Find the failure rate of the Exponential distribution

Answer:

The reliability is \( R(t) = \int_{t}^{\infty} \nu e^{-\nu x} \, dx = e^{-\nu t} \)

Failure rate, \( \phi(t) = \frac{f(t)}{R(t)} = \frac{\nu e^{-\nu t}}{e^{-\nu t}} = \nu \)  Note: \( \nu \) is a constant

The fact that the failure rate is constant is a special “lack of ageing property” of the exponential distribution.

- But often failure rates actually increase with age.
Reliability function

Which of the following could be a plot of a reliability function?

\( R(t) \): probability of surviving at least till age \( t \). i.e. that failure time is later than \( t \)

1. 
2. 
3. 
4. 

Options for \( R(t) \):
- Option 1: Decreasing curve from 1 to 0%
- Option 2: Increasing curve from 0 to 42%
- Option 3: Decreasing curve from 1 to 0%
- Option 4: Increasing curve from 0 to 25%
If we *measure* the failure rate $\phi(t)$, how do we find the pdf?

$$\phi(t) = \frac{f(t)}{R(t)} = \frac{\left(\frac{dF}{dt}\right)}{1 - F(t)} = -\frac{d}{dt} \left[ \ln(1 - F(t)) \right]$$

$$F(0) = 0 \quad \Rightarrow \ln[1 - F(t)] = -\int_0^t \phi(t') dt'$$

- can hence find $F(t)$, and hence $f(t), R(t)$

**Example**

Say failure rate $\phi(t)$ measured to be a constant, $\phi(t) = \nu$

$$\Rightarrow \ln[1 - F(t)] = -\int_0^t \nu dt = -\nu t$$

$$\Rightarrow 1 - F(t) = e^{-\nu t} \Rightarrow F(t) = 1 - e^{-\nu t} \quad \Rightarrow f(t) = \frac{dF(t)}{dt} = \nu e^{-\nu t}$$

- Exponential distribution
The Weibull distribution

- a way to model failure rates that are not constant

Failure rate: $\phi(t) = mvt^{m-1}$

Parameters $m$ (shape parameter) and $\nu$ (scale parameter)

$m = 1$: failure rate constant, Weibull=Exponential

$m > 1$: failure rate increases with time

$m < 1$: failure rate decreases with time
Failure rate: $\phi(t) = mvt^{m-1}$

$$\Rightarrow \ln(1 - F(t)) = - \int_0^t mvx^{m-1} dx = -vt^m$$

$$\Rightarrow F(t) = 1 - e^{-vt^m}$$

Reliability: $R(t) = e^{-vt^m}$

Pdf: $f(t) = \frac{dF(t)}{dt} = mvt^{m-1}e^{-vt^m}$
The End!
You may attempt as many questions as you wish, but marks will be given for the best FOUR answers only.

Time allowed: ONE hour.

Each question carries TWENTY FIVE marks. The numbers beside the questions indicate the approximate marks that can be gained from the corresponding parts of the questions.

Examination handout: Maths Dept Statistical Tables, Statistical Formulae (Engineering Statistics)
6. The time till failure of a part, $T$ years, has probability density function:

$$f(t) = kr^{-4} \quad (t > 1)$$

and is zero elsewhere, where $k$ is a constant.

(a) Find the value of $k$. [5 marks]

(b) Find the mean time till failure. [5 marks]

(c) Find the failure rate. [5 marks]

(d) Sketch a graph of the failure rate against time. What does the shape of this graph tell you? [5 marks]

[Note: as from 2011 questions are out of 25 not 20]
The time till failure of a part, $T$ years, has probability density function:

$$f(t) = kt^{-4} \quad (t > 1)$$

and is zero elsewhere, where $k$ is a constant.

(a) Find the value of $k$.

1. $\frac{1}{4}$
2. 3
3. 4
4. -3
5. $\frac{1}{3}$
The time till failure of a part, $T$ years, has probability density function:

$$f(t) = kt^{-4} \quad (t > 1)$$

and is zero elsewhere, where $k$ is a constant.

(a) Find the value of $k$.

**Answer**

\[
\int_{-\infty}^{\infty} f(t) = 1 \Rightarrow \int_{1}^{\infty} kt^{-4} \, dt = 1
\]

\[
\int_{1}^{\infty} kt^{-4} \, dt = \left[ \frac{kt^{-3}}{-3} \right]_{1}^{\infty} = 0 - \left( -\frac{k}{3} \right) = \frac{k}{3}
\]

\[
\Rightarrow k = 3
\]
The time till failure of a part, $T$ years, has probability density function:

$$f(t) = kt^{-4} \quad (t > 1)$$

and is zero elsewhere, where $k$ is a constant.

(b) Find the mean time till failure. [5 marks]

**Answer**

Know $k = 3$

$$\langle T \rangle = \int_{-\infty}^{\infty} tf(t)dt = \int_{1}^{\infty} \frac{k}{t^3} = \left[ \frac{-k}{2t^2} \right]_{1}^{\infty} = 0 - \frac{-k}{2} = \frac{3}{2}$$
The time till failure of a part, $T$ years, has probability density function:

$$f(t) = kt^{-4} \quad (t > 1)$$

and is zero elsewhere, where $k$ is a constant.

(c) Find the failure rate. [5 marks]

**Answer**

$$\phi(t) = \frac{f(t)}{R(t)}$$

$$R(t) = 1 - F(t)$$

$$F(t) = \int_0^t f(x)dx = \int_1^t \frac{k}{x^4}dx = \left[ \frac{k}{-3t^3} \right]_1^t = -\frac{k}{3t^3} - \left( -\frac{k}{3} \right) = 1 - \frac{1}{t^3}$$

$$\Rightarrow R(t) = 1 - F(t) = \frac{1}{t^3}$$

$$\Rightarrow \phi(t) = \frac{f(t)}{R(t)} = \frac{k}{t^4} \times t^3 = \frac{3}{t} \quad (t > 1), \text{ otherwise } 0$$
The time till failure of a part, $T$ years, has probability density function:

$$f(t) = kt^{-4} \quad (t > 1)$$

and is zero elsewhere, where $k$ is a constant.

(d) Sketch a graph of the failure rate against time. What does the shape of this graph tell you?

**Answer**

$\phi(t)$