7. Reliability

There are various issues in reliability. We have already looked at determining the reliability of a system if the reliabilities of its components are known. Here we are interested in the time till failure of parts. Information about failure times is needed in order to guarantee the quality of products. Typical questions include:

- what is the mean time till failure?
- what is the probability that an item fails before a specified time?

Testing

Modern products are usually expected to last for many years. Having designed a product, how do you quickly get an idea of its time till failure?

You can modify the method of testing, *accelerated life* testing:

- compressed-time testing: product is tested under the usual conditions, but more intensively (e.g. a washing machine used almost continuously)
- advanced-stress testing; product is tested under harsher conditions than it will suffer in regular use so that failure will tend to occur earlier (e.g. refrigerator motor run at a higher speed than if operating within a fridge). Need to assume a relation between time till failure under stressed conditions and under normal conditions e.g. 1 week under stress $\equiv$ 1 year under normal. This adds to the uncertainty of inferences.

Whichever method of testing is used, it is likely that you will not have time to wait until every single item has failed. How do you deal with items which are still working at the end of the test programme?

Maximum likelihood method for censored Exponential data

The data is called censored if it is incomplete, in this case we can't wait long enough for all parts to fail, so we don't know all the failure times. If a part has a constant probability of failure per unit time, we showed previously that the time till first failure is given by an Exponential distribution.

Recall that if $t_1,\ t_2,\ \ldots\ ,\ t_n$ are a random sample from an Exponential distribution with parameter ν, each time has pdf $P(t_i) = \nu e^{-\nu t_i}$. The total likelihood is the product of the independent pdf's:

$$L(\nu) \equiv P(\{t_i\}|\nu) = \nu^n \exp\left(-\nu \sum_i t_i\right).$$

Suppose that we observe parts for up to time t_0 and n_f of the items fail up to this time, with failure times $t_1,\ t_2 \ldots t_{n_f}$. The other $n_w = n - n_f$ items are still working at time t_0. For each item

$$\begin{aligned} P(T > t_0) &= P(\text{Still working at time } t_0) \\ &= \int_{t_0}^{\infty} \nu \exp(-\nu t)\,\mathrm{d}t \\ &= \left[-\exp(-\nu t)\right]_{t_0}^{\infty} = e^{-\nu t_0}. \end{aligned}$$

(as expected, this is just the Poisson probability for no events with mean νt_0). This is the contribution to the likelihood at time t_0 from non-failed parts; for the failed parts we know the failure time t_i so the total likelihood is

$$\begin{aligned} L(\nu) &= \nu^{n_f} \exp\left(-\nu \sum_{i=1}^{n_f} t_i\right) \left[e^{-\nu t_0}\right]^{n_w} \\ &= \nu^{n_f} \exp\left(-\nu \sum_{i=1}^{n} t_i\right) \end{aligned}$$

if we define $t_i = t_0$ for working parts. We can now find the maximum-likelihood estimator $\hat{\nu}$ for ν:

$$\frac{dL}{d\nu} = n_f \nu^{n_f - 1} \exp\left(-\nu \sum_{i=1}^{n} t_i\right) - \nu^{n_f} \left(\sum_{i=1}^{n} t_i\right) \exp\left(-\nu \sum_{i=1}^{n} t_i\right) = 0$$

$$\implies n_f \hat{\nu}^{n_f - 1} = \hat{\nu}^{n_f} \left(\sum_{i=1}^{n} t_i\right) \implies \hat{\nu} = \frac{n_f}{\sum_i t_i}.$$

Example: 50 components are tested for two weeks. 20 of them fail in this time, with an average failure time of 1.2 weeks. What is the mean time till failure?

Answer: $n = 50$, $n_f = 20$, $\sum_i t_i = 20 \times 1.2 + 30 \times 2 = 84$ weeks. So the estimate of the failure frequency is

$$\hat{\nu} = \frac{n_f}{\sum_i t_i} = \frac{20}{84} = 0.238/\text{week}.$$

Hence the mean time till failure is estimated to be $1/\hat{\nu} = 1/0.238 =$ 4.2 weeks.

Reliability function and failure rate (or hazard function)

Although the pdf, $f(t)$, describes the time till failure completely, it does not directly indicate either the chance of the part continuing to work for a given period of time or how the chance of failure depends on the age of the part. So we define

- **Reliability function**:

$$\begin{aligned} R(t) &= P(T > t) = \int_t^\infty f(x)dx \\ &= 1 - F(t) = \text{probability of surviving at least till age } t \end{aligned}$$

  where $F(t)$ is the cumulative distribution function.

- **Failure rate,**

$$\phi(t) = \frac{f(t)}{R(t)} = \text{rate of failure given survival till age } t.$$

Justification for the definition of $\phi(t)$

Suppose that the part has survived until time t and we want to evaluate the conditional probability that it fails before time $t + \delta t$

$$\begin{aligned} P(T \leq t + \delta t | T > t) &= \frac{P(T \leq t + \delta t \text{ and } T > t)}{P(T > t)} \\ &= \frac{1}{R(t)} \int_t^{t+\delta t} f(x)dx \approx \frac{\delta t f(t)}{R(t)} \end{aligned}$$

In a time δt we expect rate $\times\ \delta t = \phi(t)\delta(t)$ failures, so

$$\frac{\delta t f(t)}{R(t)} = \phi(t)\delta(t) \qquad \Longrightarrow \qquad \phi(t) = f(t)/R(t).$$

Example: Find the failure rate of the Exponential distribution.

Solution The reliability is

$$R(t) = \int_t^\infty \nu e^{-\nu x} dx = e^{-\nu t}$$

as before. Failure rate, $\phi(t) = \frac{\nu e^{-\nu t}}{e^{-\nu t}} = \nu$, a constant.

The fact that the failure rate is constant is a special "lack of ageing property" of the exponential distribution. Some components behave like this, but for many the failure rate increases with age. We need a more flexible model to describe failure times than the Exponential.

Deducing the distribution from the failure rate

$$\phi(t) = \frac{f(t)}{R(t)} = \frac{\frac{dF}{dt}}{1 - F(t)} = -\frac{d}{dt}\left[\ln(1 - F(t))\right].$$

$$\Longrightarrow \ln\left[1 - F(t)\right] = -\int \phi(t)dt.$$

Since $F(0) = 0$,

$$\ln[1 - F(t)] = -\int_0^t \phi(x)dx$$

This can be used to find $F(t)$ and hence $R(t)$ and $f(t)$ from $\phi(t)$.

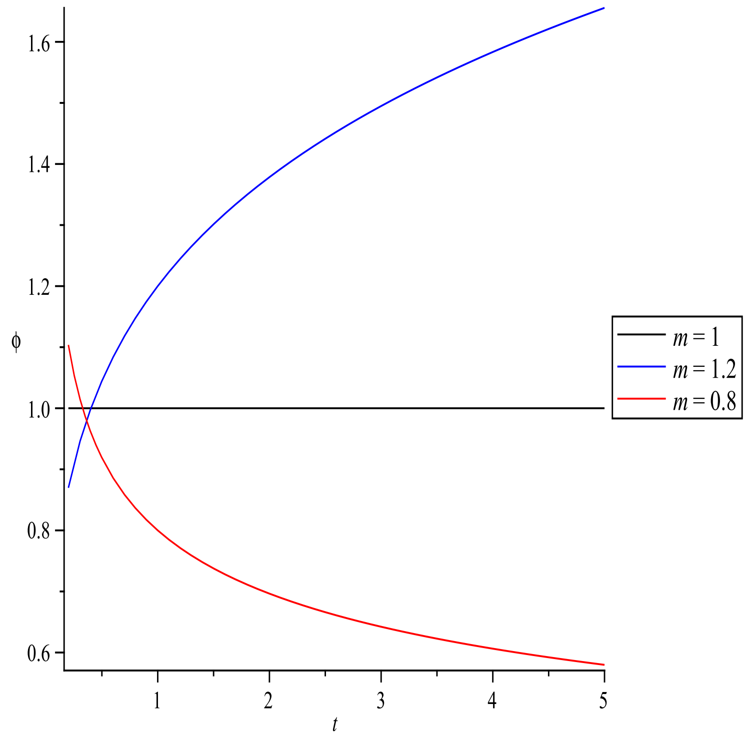
Example Say $\phi(t) = \text{constant} = \nu$. This gives

$$\ln(1 - F(t)) = -\int_0^t \nu dx = -\nu t \implies F(t) = 1 - e^{-\nu t}.$$

Then we can get $f(t)$ using $f(t) = dF/dt = \nu e^{-\nu t}$, the exponential distribution, as expected.

The Weibull distribution

The Weibull distribution has two parameters, m and ν, both positive. Failure rate is given by $\phi(t) = m\nu t^{m-1}$ for $t > 0$. So for $m = 1$ the rate is constant, for $m > 1$ the rate increases with time (ageing), and for $m < 1$ the rate decreases with time (defective components are gradually weeded out). The distribution can therefore be used to model failures where the rate is not constant in time.



Using the failure rate we have

$$\ln(1 - F(t)) = -\int_0^t m\nu x^{m-1} dx = -\nu t^m \implies F(t) = 1 - e^{-\nu t^m}.$$

So the reliability is $R(t) = e^{-\nu t^m}$ and

$$f(t) = m\nu t^{m-1} e^{-\nu t^m}.$$

The m parameter is a shape parameter and ν is a scale parameter (as for the exponential). For $m = 1$ the Weibull distribution gives the exponential distribution as a special case.

For a Weibull distribution $\ln[-\ln(R(t))] = \ln(\nu) + m\ln(t)$ so plotting $\ln[-\ln(R(t))]$ against $\ln(t)$ should give a straight line, approximately.

Estimating m and ν

A graphical method is given in Chatfield. A better method, which copes with the case of censored data, is to use maximum likelihood.

Suppose that we observe parts for up to time t_0 and n_f of the items fail up to this time, with failure times t_1, t_2... t_{n_f}. The other $n_w = n - n_f$ items are still working at time t_0. If the distribution is Weibull the likelihood is given by:

$$\begin{aligned} L(\nu, m) &= \left(\prod_{i=1}^{n_f} m\nu t_i^{m-1} e^{-\nu t_i^m} \right) \left[e^{\nu t_0^m} \right]^{n_w} \\ &= m^{n_f} \nu^{n_f} \left(\prod_{i=1}^{n_f} t_i \right)^{m-1} \exp\left(-\nu \sum_{i=1}^{n} t_i^m \right). \end{aligned}$$

The log-likelihood is

$$\ln L(\nu, m) = n_f \ln m + n_f \ln \nu + (m-1) \sum_{i=1}^{n_f} \ln(t_i) - \nu \sum_{i=1}^{n} t_i^m$$

hence the maximum likelihood is given by

$$\frac{\partial \ln L}{\partial \nu} = \frac{n_f}{\hat{\nu}} - \sum_{i=1}^{n} t_i^m \implies \hat{\nu} = \frac{n_f}{\sum_{i=1}^{n} t_i^m}.$$

The maximum w.r.t. m generally has to be found numerically.