Early Universe : Problem Sheet 3

Deadline: Week 12, Thursday 4th May at 12:00. Approximate marks in [].

Hand in at the MPS School Office. Solutions submitted up 24 hours late will be attract a penalty of 5%; no solutions will be accepted more than 24 hours late.

1 Gravitational waves [18]

Consider harmonic expansion of the gravitational wave distribution with $H_\pm$ of either helicity.

(i) Starting from the field equation in the notes, show that $u \equiv aH_\pm$, satisfies the equation

$$u'' + \left(k^2 - \frac{a''}{a}\right) u = 0.$$

(ii) Show that for $k \gg \mathcal{H}$ (specifically $a''/a \ll k^2$) that the amplitude of the sub-horizon tensor perturbations decays as $1/a$.

(iii) Find the solution for $H_\pm$ in radiation domination [now for any $k/\mathcal{H}$], with $H_\pm = 1$ at $\eta = 0$ (i.e. the transfer function).

(iv) Assume the spectrum of primordial gravitational waves is scale-invariant, so at $\eta = 0$ we have $\mathcal{P}_T(k, 0) = A_T$.

(a) Find the power spectrum at matter-radiation equality (in terms of the conformal time η_{eq}), approximating the universe as radiation dominated.

(b) Sketch it as a function of $\ln k$, labelling any relevant scales.

2 Sound speed [15]

The sound speed can be taken to be $c_s^2 = dp/d\rho$.

(i) Demonstrate, using their redshift evolution, that if one has a fluid of photons (γ) and baryons (b, approximated as being pressureless), then

$$c_s^2 \;=\; \frac{1}{3}\left(\frac{3}{4}\frac{\rho_b}{\rho_\gamma} + 1\right)^{-1}.$$

(ii) What is the sound speed at recombination (at redshift z_*)? [assume a flat universe with $\Omega_b h^2 = 0.02$, $z_* = 1000$, $T_0 = 2.7K$]

(iii) The amplitude of the sub-horizon radiation perturbations at recombination ($\eta = \eta_*$) have an approximately sinusoidal dependence in wavenumber k. If the baryon density is increased (keeping η_* and the background evolution fixed), would you expect the k-spacing of the oscillations to increase, stay the same, or decrease, and why?

3 Massive neutrinos [15]

Consider a flat universe ($\Omega_{tot} = 1$) filled with cold, pressureless matter and massive neutrinos (which make up a fraction $\Omega_\nu \ll 1$ of the critical density). The background dynamics of the universe in this situation can be approximated as matter dominated.

(i) Write down the coupled system of equations for the cold matter and neutrino density perturbations on sub-horizon scales

(ii) Show that, on scales smaller than the neutrino Jeans length (so that neutrino perturbations can be neglected compared to dark matter), perturbations in the remaining cold component grow as $\Delta_c \propto \eta^\alpha$, where $\alpha = (\sqrt{25 - 24\Omega_\nu} - 1)/2$. Hence find how Δ_c grows with time t.

4 Matter power spectrum [17]

1. Statistical homogeneity implies that the real-space correlation between a density at two points (at fixed time) is the same under a spatial translation, i.e. $\langle \Delta(\mathbf{x})\Delta(\mathbf{x}')\rangle = \langle \Delta(\mathbf{x}+\mathbf{c})\Delta(\mathbf{x}'+\mathbf{c})\rangle$ where $\mathbf{c}$ is any constant vector.
   (a) Show that statistical homogeneity implies that $\langle \Delta(\mathbf{k})\Delta(\mathbf{k}')\rangle \propto \delta(\mathbf{k}+\mathbf{k}')$.
   (b) What additional constraint does statistical isotropy impose?
2. (a) Sketch the expected shape of the linear-theory matter power spectrum as a function of $\log(k)$
   (b) Explain in your own words why it has the rough shape that it does.
   (c) Indicate how the spectrum would change if n_s, the spectral index of the primordial curvature power spectrum is increased.
3. It is often useful to consider the density perturbation smoothed over some scale, for example a Gaussian smoothing with some scale σ:

$$\Delta_\sigma(\mathbf{x}') \equiv \int d^3\mathbf{x} \frac{e^{-(\mathbf{x}-\mathbf{x}')^2/2\sigma^2}}{(2\pi\sigma)^{3/2}} \Delta(\mathbf{x}). \quad (1)$$

   (a) Calculate the power spectrum P_{Δ_σ} of Δ_σ in terms of P_Δ [*hint*: convolution theorem].
   (b) Sketch roughly the expected form of P_{Δ_σ} compared to P_Δ.

5 Growth factor and velocity power spectrum [12]

On scales well within the horizon but larger than the Jeans length, at a given time we can define a (k-independent) growth factor g where

$$g \equiv \frac{d(\ln \Delta_m)}{d(\ln a)}. \tag{2}$$

(i) (a) What is g during matter domination?

(b) What is g in the far future if Λ (cosmological constant) then dominates?

(ii) Calculate the power spectrum P_θ of $\theta \equiv \boldsymbol{\nabla} \cdot \mathbf{v}_m / \mathcal{H}$ in terms of g and P_{Δ_m}.

6 Photon redshift in a perturbed universe [23]

Consider the propagation of photons (null geodesics) in a perturbed flat universe in the conformal Newtonian gauge (CNG) with metric

$$ds^2 = a(\eta)^2 \left[(1+2\Psi)d\eta^2 - (1-2\Phi)d\mathbf{x}^2\right].$$

Normalize the affine parameter λ so that a photon's 4-momentum is given by $p^\mu = dx^\mu/d\lambda$. Working to linear order in the perturbations Ψ and Φ:

1. If an observer at rest in the CNG has 4-velocity $u^\nu = (1-\Psi)\delta_{0\nu}/a$ show that $u_\nu = a(1+\Psi)\delta_{0\nu}$ and hence that $u_\nu u^\nu = 1$. (remember we are only working to first-order in Ψ)

2. Show that if a photon with 4-momentum p^μ has an energy $E = p^\nu u_\nu$ for an observer with 4-velocity u^ν then
$$p^0 = \frac{d\eta}{d\lambda} = E(1-\Psi)/a.$$

3. Calculate the $\Gamma^0_{\mu\nu}$ Christoffel symbols, and show hence that the 0 component of the geodesic equation can be written
$$\frac{dp^0}{d\lambda} + \left(\frac{a'}{a} + \Psi'\right) p^0 p^0 + 2p^0 p^i \frac{\partial \Psi}{\partial x^i} - \left(\frac{a'}{a}(1-2\Psi) - \Phi'\right) \frac{g_{ij}}{a^2} p^i p^j = 0$$
where i, j run over the spatial indices and primes denote conformal time derivatives. Using $p^i \partial/\partial x^i = d/d\lambda - p^0 \partial/\partial\eta$ then show this can be written as
$$\frac{d[aE(1+\Psi)]}{d\lambda} = E^2\left(\Psi' + \Phi'\right)$$
(*hint:* use the fact that $\epsilon \equiv aE$ is constant to zeroth order, writing p^0 in terms of ϵ).

4. Hence if the energy observed by a observer (at rest in the CNG) today is E_0 at conformal time η_0, show that
$$E(\eta) = \frac{E_0}{a(\eta)}\left[1 + \Psi_0 - \Psi(\eta) - \int_\eta^{\eta_0} d\eta(\Psi' + \Phi')\right]$$
where the integral is along the photon path. Explain the relevance of this result for the observed CMB anisotropies.