Advanced Cosmology : Problem Sheet 2

Deadline: Week 9, Monday 23rd March, at 12:00. Approximate marks in [], with this sheet marked as a total out of 77.

Hand in at the MPS School Office. Solutions submitted up 24 hours late will be attract a penalty of 5%; no solutions will be accepted more than 24 hours late.

1 Scalar field Lagrangian [12]

The Lagrangian for a homogeneous scalar field is

$$\mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

Derive the field equation using the Euler-Lagrange equations with the action $\int d^4x\sqrt{-g}\mathcal{L}_\phi$ in a flat FRW universe, and show that it is consistent with the density evolution equation. Change variables to conformal time to show that

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_{,\phi} = 0,$$

where a prime denotes $d/d\eta$, η is the conformal time and $\mathcal{H} = aH$.

2 Power law inflation [20]

An inflation model *power law inflation* has a potential given by

$$V(\phi) = V_0 \exp\left\{-\sqrt{\frac{2}{\lambda}}\frac{\phi}{M_P}\right\},$$

where V_0 and λ are a priori free parameters of the model. Assume that the universe is flat and that the energy density is dominated by the inflaton.

(i) Write down the relations for the density and pressure (ρ and P), the equation of motion and the slow-roll equations.

(ii) Using the slow-roll approximation compute solutions for $\phi(t)$, $w(t)$, $a(t)$ and the slow-roll parameters ϵ_V and η_V. When does inflation end?

(iii) What conditions have to be satisfied for "successful" inflation in this model?

3 Power spectrum from quantum fluctuations [20]

A scalar field perturbation operator in harmonic space $\hat{\delta\phi}$ is expanded as

$$\hat{\delta\phi}(\mathbf{k},\eta) = w(k,\eta)\hat{a}(\mathbf{k}) + w^*(k,\eta)\hat{a}^\dagger(-\mathbf{k}),$$

where $w(k,\eta)$ is a solution of the field equation

$$w'' + 2\mathcal{H}w' + k^2 w = 0.$$

(i) By expanding $\hat{\delta\phi}$ and $\hat{\pi}_{\delta\phi}$ into Fourier modes show that

$$[\hat{\delta\phi}(\mathbf{x},\eta), \hat{\pi}_{\delta\phi}(\mathbf{x}',\eta)] = i\delta(\mathbf{x}-\mathbf{x}')$$

if we require that

$$a^2(ww^{*\prime} - w'w^*) = i$$

and

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = \delta(\mathbf{k}-\mathbf{k}'), \qquad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 \qquad [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0.$$

(ii) If the Hubble rate H is constant solve for the comoving Hubble rate $\mathcal{H}$ as a function of conformal time (with $-\infty < \eta < 0$).

(iii) If H is constant show that

$$w(k,\eta) = \frac{H(k\eta - i)}{\sqrt{2k^3}} e^{-ik\eta}$$

solves the field equation and the normalization constraint. Sketch $|w|$ as a function of conformal time around the time a given k leaves the horizon.

4 Scales and times [13]

1. Assume naively that inflation happened with $H \sim H_i \sim$ constant and ended suddenly giving a hot big bang (radiation dominated universe) that started with $H = H_i$ when $a \sim 10^{-15}$. Does inflation increase or decrease the age of the universe between scale factor $a \sim 10^{-40}$ and $a \sim 10^{-15}$, and by roughly how much? (compared to a pure hot big bang model starting at $a = 0$)

2. Perturbations undergo different evolution depending on whether their (comoving) wavenumber k is smaller or larger than the (comoving) Hubble scale $\mathcal{H} = aH$.

   (i) Compute the evolution of the comoving Hubble length with time during inflation ($w \approx -1$), radiation domination ($w = 1/3$) and matter domination ($w = 0$).

   (ii) Sketch the behavior of the comoving Hubble length with time (perhaps using logarithmic scales). What does this tell us about the evolution of a perturbation with fixed comoving wavenumber, k?

5 ϕ^4 inflation [12]

Consider the Inflation potential $V = \lambda\phi^4$, where λ is the self coupling. Assume that the field rolls towards $\phi = 0$ from the positive side.

(i) Calculate the value of ϕ where each of the slow-roll conditions break down.

(ii) Assuming that inflation ends when $\epsilon = 1$, calculate the number of e-folds of inflation that occur for an initial value of ϕ_i.

(iii) Assuming that the scales of interest left the horizon 55 e-foldings before the end of inflation, calculate the scalar amplitude A_s and spectral index. Use the Planck result for A_s to put an order of magnitude value on λ.