Early Universe : Unassessed problem 2

1 Geodesics and the FRW solution

This question is about geodesics, and then using a trick to calculate the Christoffel symbols and hence the Friedmann equations. Geodesics are paths through spacetime with extremal 'length', i.e. they extremize

$$S = \int ds = \int d\lambda \frac{ds}{d\lambda} = \int d\lambda L$$

where from the metric

$$L \equiv \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$

1. For a timelike geodesic with $\lambda = \tau$ the proper time, show that $x^\alpha(\tau)$ that extremizes S also extremizes

$$S_2 = \int d\tau L_2$$

with $L_2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$. Do we expect this to be a minimum or maximum?

2. Apply the Euler-Lagrange equations

$$\frac{d}{d\tau} \left(\frac{\partial L_2}{\partial (dx^\alpha / d\tau)} \right) = \frac{\partial L_2}{\partial x^\alpha}$$

to show that the general solution for the the path $x^\alpha(\tau)$ along the geodesic is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0.$$

3. Write down L_2 in the flat FRW metric

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j$$

and then apply the Euler-Lagrange equations with $x^\alpha = t$ find an equation for $d^2 t / d\tau^2$. By comparing this with Eq. 2 read off all the components α, β of $\Gamma^t_{\alpha\beta}$. (this is a handy trick for quickly calculating several Christoffel symbols)

4. Similarly calculate all the other non-zero Christoffel symbols. Explain why the Ricci tensor is expected to be diagonal with $R_{ij} \propto \delta_{ij}$, and calculate its non-zero components.

5. Calculate the Ricci scalar, and hence the Einstein equations for the FRW metric with the energy-momentum tensor

$$T^\nu_\mu = \text{diag}\ (\ \rho\ , -P\ , -P\ , -P\ ).$$