Early Universe : Problem Sheet 1

Deadline: Week 5, 4th March, at 12:00. Approximate marks in [], with this sheet marked as a total out of 85. Hand in at the MPS School Office. Solutions submitted up 24 hours late will be attract a penalty of 5%; no solutions will be accepted more than 24 hours late.

Q 1 is relativity revision (related to cosmology - the result is needed later in the course); you could start work on this immediately.

1 The spatial curvature [17]

Calculate the 3-curvature, the Ricci scalar of the spatial part of the metric at fixed time:

$$dl^2 = a^2 \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right].$$

(first calculate the Christoffel symbols, then the curvature tensors). Note that at fixed time, there are no time derivatives.

2 Extra relativistic species [20]

According to the standard assumptions, there are three species of (massless) neutrinos. In the temperature range of $1\text{MeV} < T < 100\text{MeV}$, the density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and three neutrinos all of which were in thermal equilibrium.

1. Neglecting any change in the degrees of freedom at $T > 100\text{MeV}$, show using the Friedmann equation for a flat radiation-dominated universe $H^2 = 8\pi G\rho_R/3$ that for temperatures $T > 1MeV$ the time since the start of the hot big bang is given by

$$t(T) = \left(\frac{A}{g_*}\right)^{1/2} \frac{M_P}{T^2}$$

where $M_P \equiv \sqrt{\hbar c/(8\pi G)}$ is the reduced Planck mass, and A is a constant that you should give explicitly. What is g_*? Put in $\hbar$, c and k_B factors give a result in standard rather than natural units. How long did it take from the big bang for the temperature to fall to $T = 1\text{MeV}$? [Give the result in seconds]. [7,2]

2. How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming? [3]

3. What would be the density of the universe (in kg/m^3 units) when $T = 1\text{MeV}$ under the standard assumptions, and what would it be if there were one other species of massless neutrino? What is the temperature in Kelvin at $T = 1\text{MeV}$, and what is the redshift? [4,3]

4. What approximation have you made about the electrons and positron velocities, and is it reasonable? [1]

3 Freeze out of muons [16]

Muons μ^- are essentially identical to electrons, except that they are heavier ($m_\mu = 106MeV$); other than that, they also have the same charge and spin as the electron, and there is an antimuon μ^+ analogous to the positron.

1. What is the value of the effective g_* for muons when they are relativistic? [2]
2. When the temperature T is a little above 106MeV, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the total effective g_*? [5]
3. As T falls below 106MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting a denote the FRW scale factor, by what factor does the quantity aT increase when the muons disappear? [In case you worry about it, ignore pions and QCD] [9]

4 CMB blackbody and μ-distortions [16]

The distribution function of photons in a homogeneous and isotropic photon gas in kinetic equilibrium is

$$f_\gamma(E,T) = \frac{2}{(2\pi)^3}\frac{1}{e^{(E-\mu_\gamma)/T}-1}.$$

Consider the homogenous universe well after electron-positron annihilation is complete and ignore the very small effect of baryons.

1. At high temperatures $T \gg T_c \sim 0.5$keV double Compton scattering ($e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$) happens frequently in equilibrium. In this case explain why $\mu_\gamma = 0$. [2]
2. At lower temperatures $T \ll T_c$ double Compton scattering no longer happens, and in general μ_γ can be non-zero. As the gas cools below T_c it initially maintains its thermal distribution with $\mu_\gamma = 0$. If a small amount of energy is then injected into the photon gas to give an increase in the energy density by ϵ, show by doing a first order series expansion in δT and μ_γ that after (rapid) kinetic thermalization

   $$\epsilon \approx \frac{T^4}{\pi^2}\int_0^\infty \frac{e^x x^3 dx}{(e^x-1)^2}\left[\frac{\mu_\gamma}{T} + x\frac{\delta T}{T}\right],$$

   where the temperature is changed by δT and the chemical potential is changed by μ_γ (from zero). You can assume that $|\mu_\gamma/T| \ll 1$, $|\delta T/T| \ll 1$. [7]
3. If the energy injection increases the energy density *without* changing the number density of photons n_γ, show that after kinetic thermalization

   $$\frac{\mu_\gamma}{T} \approx \frac{\epsilon}{\rho_\gamma}\frac{CX_3}{X_3^2 - X_2X_4}$$

where X_k and C are defined to be the values of the integrals

$$X_k \equiv \int_0^\infty \frac{x^k e^x dx}{(e^x-1)^2} \qquad C \equiv \int_0^\infty \frac{x^3 dx}{e^x - 1}.$$

[This shows that processes depositing energy at $T < 0.5$keV can give rise to a "μ-distortion" in the CMB, i.e. a not-exactly blackbody spectrum.] [7]

5 Neutrino mass [16]

At least two neutrinos are thought to have a small mass, but small enough that in the early universe the neutrinos are still very relativistic. Assuming zero neutrino chemical potential:

1. The equilibrium distribution function at temperature T for a single neutrino species in the limit in which the mass can be neglected, using natural units where $k_B = c = \hbar = 1$, is

$$f_\nu(p, T) = \frac{g_\nu}{(2\pi)^3} \frac{1}{e^{p/T}+1}$$

   What is the meaning of f_ν and p here, and what is the value of g_ν? [2]

2. After a massive neutrino has completely decoupled at temperature T_D and scale factor a_D, show that the energy density in these neutrinos is given by

$$\rho_\nu = \frac{T_\nu^4}{\pi^2} \int_0^\infty \frac{x^2 dx \sqrt{m_\nu^2/T_\nu^2 + x^2}}{e^x + 1}$$

   where $T_\nu \equiv T_D a_D / a$ [assume the neutrino was highly relativistic when it decoupled, so $m_\nu / T_D \ll 1$ is negligible]. [5]

3. By considering a series expansion for small m_ν/T_ν show that if there is a massless neutrino with energy density $\rho_{\nu 0}$, for a nearly-relativistic massive neutrino with mass m_ν

$$\rho_\nu \approx \rho_{\nu 0}\left(1 + \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right)$$

   to leading order in m_ν/T_ν. You can assume all the neutrinos were in thermal equilibrium before they decoupled. [6]

4. The effect of massive neutrinos can be seen in the linear CMB anisotropies if they are massive enough to affect the background evolution before recombination, e.g. when ρ_ν is significantly larger than $\rho_{\nu 0}$. Approximately what is the lightest neutrino (in electron volts) that has an observable effect on the linear CMB anisotropies? [3]