Early Universe : 2016 Open Note Test

Answer TWO questions out of three, each of which is marked out of 20.

Time allocated: 1Hr 30 minutes.

1. Assume a single-field slow-roll inflation model ending in a hot big bang, and approximate the potential V (and hence Hubble parameter H_i) as constant during inflation.

   (a) Explain briefly what is meant by the *Horizon problem* in a perturbed hot-big-bang cosmology with no inflation. [3]

   (b) Approximate the reheating process as a sudden transition between inflation and a radiation-dominated universe (in thermal equilibrium) with initial temperature T_*. Using the fact that the energy density just before and after reheating should be the same, relate T_* to V in terms of g_*^*, the effective degrees of freedom of the radiation just after reheating. [2]

   (c) A linear perturbation mode with comoving wavenumber k leaves the horizon at scale factor a_k during inflation when $k \approx a_k H_i$. The number of e-foldings between when a mode k leaves the horizon and reheating at scale factor a_* is defined as

   $$N_k \equiv \ln(a_*/a_k).$$

   Using conservation of entropy after the hot big bang, with entropy $S \propto g_{*S}(aT_\gamma)^3$ (where T_γ is the photon temperature), show that in terms of the photon temperature today T_γ^0,

   $$N_k \approx \ln\left(A(g_*^*)^{-\frac{1}{12}} \frac{T_\gamma^0}{k} \frac{V^{1/4}}{M_P} \right),$$

   for some constant A that you should give explicitly in terms of g_{*S}^0 (the effective degrees of freedom for entropy density today). Here we use natural units with $\hbar = c = k_B = 1$ and M_P is the reduced Planck mass. [9]

   (d) Evaluate g_{*S}^0. Then, assuming $g_*^* = g_{S*}^* \approx 106.75$ and $V^{1/4} = 10^{16}\text{GeV}$, how many e-foldings are required to solve the horizon problem for a perturbation with $k = 0.001\text{Mpc}^{-1}$? [note that $k_B\text{Mpc}/(\hbar c) = 1.3475 \times 10^{25} K^{-1}$] [4]

   (e) Defining a temperature during inflation of $T_{\text{inf}} \equiv \frac{H_i}{2\pi}$, if inflation ends at potential V_*, by what factor does the temperature change at reheating? Evaluate this for $g_*^* \approx 106.75$, $V_*^{1/4} = 10^{16}\text{GeV}$. [2]

2. Consider a single-field slow-roll inflation model with potential $V(\phi)$ and slow-roll parameters defined by

$$\epsilon_V \equiv \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta_V \equiv M_P^2 \frac{V_{,\phi\phi}}{V},$$

where M_P is the reduced Planck mass.

(a) Explain what is meant by the *number of e-foldings* N of inflation. Show that if ϕ increases during inflation and inflation ends at $\phi = \phi_{\rm end}$ that

$$N(\phi) = M_P^{-1}\int_{\phi}^{\phi_{\rm end}} \frac{d\phi'}{\sqrt{2\epsilon_V(\phi')}}$$

[4]

(b) Assuming that $\phi > 0$ and ϕ increases during inflation, and that $(V_{,\phi}/V)^2$ monotonically increases during inflation (so that $\epsilon_V(\phi) < \epsilon_V(\phi_{\rm end})$), by considering a lower limit for N show that at any point during inflation

$$2\epsilon_V(\phi) < \frac{1}{N(\phi)^2}\left(\frac{\Delta\phi}{M_P}\right)^2$$

where $\Delta\phi = \phi_{\rm end} - \phi$. [5]

(c) Using this result show that the tensor-scalar ratio r is bounded by

$$r < A\left(\frac{60}{N}\right)^2\left(\frac{\Delta\phi}{M_P}\right)^2$$

where A is a numerical constant that you should give. [2]

(d) A special case of such an inflation model is 'hilltop inflation'

$$V(\phi) = V_0 - \frac{m^2}{2}\phi^2,$$

with ϕ starting near zero and increasing during inflation. Using the approximation that for most of the evolution $\frac{m^2}{2}\phi^2 \ll V_0$, find $N(\phi)$ and hence show that for this model

$$r \approx B(n_s - 1)^2 e^{(n_s-1)N}\left(\frac{\phi_{\rm end}}{M_P}\right)^2,$$

where B is a constant that you should determine. [7]

(e) If $\phi_{\rm end} < M_P$, using rough values for current observational constraints (e.g. $n_s \approx 0.96$, $40 < N < 60$) give an approximate numerical upper limit on the allowed value of r for hilltop inflation. [2]

3. Scalar dark matter perturbations on sub-horizon scales in a flat universe evolve approximately according to

$$\Delta_c' + \boldsymbol{\nabla} \cdot \mathbf{v}_c = 0 \quad (1)$$

$$\mathbf{v}_c' + \mathcal{H}\mathbf{v}_c = -\boldsymbol{\nabla}\Phi, \quad (2)$$

where primes denote derivatives with respect to conformal time, $\mathcal{H} = aH$ is the conformal Hubble parameter, $\mathbf{v}_c$ is the velocity of the dark matter fluid and $\Delta_c = \delta\rho_c/\rho_c$. The Poisson equation relates the potential to the densities as

$$\boldsymbol{\nabla}^2\Phi = \frac{\kappa}{2}a^2\delta\rho = \frac{\kappa}{2}a^2\sum_i \delta\rho_i.$$

Throughout this question approximate the matter as being only dark matter (i.e. neglect baryons), so that the sum runs only over radiation and dark matter, and the total energy density is $\rho = \rho_c + \rho_R$ where ρ_c is the density in dark matter and ρ_R is the density in radiation (neglect dark energy).

(a) Explain what it means to describe a perturbation mode as "sub-horizon", and for sub-horizon modes how important terns like $\boldsymbol{\nabla}^2\Phi$ are compared to terms like $\mathcal{H}^2\Phi$. [3]

(b) Show that if radiation perturbations can be neglected

$$\Delta_c'' + \mathcal{H}\Delta_c' - \frac{3y}{2(1+y)}\mathcal{H}^2\Delta_c = 0$$

where $y \equiv \rho_c/\rho_R$. [5]

(c) Show that

$$\frac{d\mathcal{H}}{dy} = -\frac{\mathcal{H}}{2y}\frac{2+y}{1+y}$$

and hence using the result of part (b) that

$$\frac{d^2\Delta_c}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\Delta_c}{dy} - \frac{3}{2y(1+y)}\Delta_c = 0.$$

[10]

(d) What is the growing solution at late times when $y \gg 1$? [2]