Early Universe : 2014 Open Note Test

Answer TWO questions out of three, each of which is marked out of 20.
Time allocated: 1Hr 30 minutes.

1. At least two neutrinos are thought to have a non-zero mass, but small enough that in the early universe the neutrinos are still very relativistic. Assuming zero neutrino chemical potential:

   (a) The equilibrium distribution function at temperature T for a single massless neutrino species, using natural units where $k_B = c = \hbar = 1$, is

   $$f_\nu(p, T) = \frac{g_\nu}{(2\pi)^3} \frac{1}{e^{p/T} + 1}$$

   What is the physical meaning of f_ν and p here, and what is the value of g_ν? [3]

   (b) Now consider a single massive neutrino species. After the neutrinos have completely decoupled at temperature T_D and scale factor a_D, explain the form distribution function takes and show that the energy density in these neutrinos is then given by

   $$\rho_\nu = A T_\nu^q \int_0^\infty \frac{x^2 dx \sqrt{1 + x^2 T_\nu^2 / m_\nu^2}}{e^x + 1},$$

   where $T_\nu \equiv T_D a_D / a$ [assume the neutrino was highly relativistic when it decoupled, so $m_\nu / T_D \ll 1$ is negligible], and A and q are constants that you should determine in terms of the neutrino mass. [7]

   (c) By considering a series expansion for large m_ν / T_ν (hence small T_ν / m_ν) show that for a sub-relativistic massive neutrino with mass m_ν

   $$\rho_\nu \approx n_\nu m_\nu \left(1 + \alpha \frac{T_\nu^2}{m_\nu^2}\right)$$

   to leading order in T_ν / m_ν, where n_ν is the number density and the constant α should be given explicitly in terms of Riemann Zeta functions (ζ). You can assume all the neutrinos were in thermal equilibrium before they decouple. Note that

   $$\int_0^\infty dx \frac{x^p}{e^x + 1} = p! \left(1 - \frac{1}{2^p}\right) \zeta(p + 1).$$

   [10]

2. Consider a single scalar field model of inflation, with potential

   $$V(\phi) = \alpha \left(\frac{\phi}{M_P}\right)^\beta,$$

   where M_P is the reduced Planck mass and α and β are unknown parameters (not necessarily integer), with $\alpha > 0$ and $\beta > 0$. Assume slow-roll inflation with slow-roll parameters defined by

   $$\epsilon_V \equiv \frac{M_P^2}{2} \left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta_V \equiv M_P^2 \frac{V_{,\phi\phi}}{V}.$$

   (a) Calculate the slow roll parameters as a function of α, β and ϕ [2]

   (b) Assuming slow-roll, find the number of e-foldings N if the inflaton field starts at $\phi = \phi_i$ and ends when $\epsilon_V = 1$. [5]

   (c) Use the slow-roll approximation result for n_s to show that β is then given in terms of N and n_s by

   $$\beta = -4 \frac{1 + (n_s - 1)N}{n_s + 1}.$$

   [6]

(d) Using this result and the slow-roll result for the tensor/scalar ratio r to find an expression for N in terms of β and r. Hence show that from observed values of n_s and r, the parameter β can be determined using

$$\beta \propto \frac{r}{r + 8(n_s - 1)},$$

where you should give the constant of proportionality explicitly. [6]

(e) The BICEP2 experiment recently measured the tensor-to-scalar ratio r to be in the range $0.11 < r < 0.22$, and Planck gives a scalar spectral index $n_s = 0.96$ to good accuracy. What is the allowed range for β? [1]

3. (a) Sketch the late-time comoving matter power spectrum as a function of $\ln(k)$ (with k in comoving inverse Megaparsecs) assuming a scale invariant primordial power spectrum of curvature perturbations. Label the slope on large-scales (low k), region of baryon acoustic oscillations, and the explain briefly why the slope changes significantly with k. [7]

(b) If the comoving angular diameter distance to last-scattering turned out to be larger than thought (e.g. due to curvature), would the frequency of the baryon oscillations (in comoving wavenumber k) inferred from CMB observations become smaller or larger, or stay the same? [3]

(c) Observations are made of the number density of galaxies over a thin redshift shell at redshift z. The observed number density in direction $\hat{\mathbf{n}}$ is then given by

$$N(\hat{\mathbf{n}}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} n(z, \mathbf{k}) e^{i\chi_z \mathbf{k}\cdot\hat{\mathbf{n}}},$$

where $n(z, \mathbf{k})$ is the Fourier transform of the galaxy number density perturbation at redshift z, and χ_z is the radial comoving distance to redshift z. Assume also that

$$n(z, \mathbf{k}) = n(z)\left[1 + b\Delta(z, \mathbf{k})\right],$$

where b is a constant and Δ is the fractional matter density perturbation. Considering only late times so that $\Delta(z, \mathbf{k}) = g(z)\Delta(0, \mathbf{k})$, show that the angular power spectrum of N is given (for $l > 0$) by

$$\langle N_{lm} N^*_{l'm'} \rangle = A\delta_{ll'}\delta_{mm'}[g(z)n(z)]^2 \int d\ln k\, \mathcal{P}_\Delta(0, k)\left[j_l(k\chi_z)\right]^2,$$

where $\mathcal{P}_\Delta(z, k)$ is the matter power spectrum with wavenumber $k = |\mathbf{k}|$, and A is a constant that you should determine. You may find the Rayleigh plane-wave expansion useful:

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(kx) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{x}})$$

where $x = |\mathbf{x}|$, $\hat{\mathbf{x}} = \mathbf{x}/x$ and $\hat{\mathbf{k}} = \mathbf{k}/k$. [10]