Early Universe : 2013 Open Note Test

Answer TWO questions out of three, each of which is marked out of 20.
Time allocated: 1Hr 30 minutes.

1. In a homogeneous and isotropic universe the distribution function for a particle A of mass m is $f_A(p,t)$. In terms of this the density and pressure of a fluid of the particles is given by

$$\rho = 4\pi \int_0^\infty dp\, p^2 E f_A(p,t) \qquad P = \frac{4\pi}{3} \int_0^\infty dp\, \frac{p^4}{E} f_A(p,t)$$

where E is the particle energy. The distribution function obeys the Boltzmann equation $\hat{L}[f_A(p,t)] = C_A(p)$, where C_A is a collision/decay term. For an FRW universe

$$\hat{L}[f_A(p,t)] = \left[E\frac{\partial}{\partial t} - Hp^2 \frac{\partial}{\partial E}\right] f_A(p,t)$$

where H is the Hubble parameter.

(a) Describe briefly in your own words what is meant by *thermal equilibrium*, and explain why and when *decoupling* of particular particle species might happen in the early universe. What happens to the momentum of a particle in an expanding FRW universe after it has completely decoupled? [5]

(b) If there were an extra completely non-interacting relativistic particle present in the early universe that doubled the total relativistic energy density, explain qualitatively how you would expect the redshift of (standard) neutrino decoupling to change. [3]

(c) By integrating the Boltzmann equation appropriately over momentum show that

$$\dot{\rho}_A + 3H(\rho_A + P_A) = X$$

where you should give the right-hand side X in terms of an integral of $C_A(p)$. [*Hint:* this is similar to the way we derived the number density equation for $\dot{n}_A$ in lectures.] [8]

(d) If X is non-zero, how can this be consistent with the energy-conservation equation? If $C_A(p)$ is positive because particle B decays into particle A, and this is the only interaction between A and B and anything else (except gravity), what is the equation for $\dot{\rho}_B$? [4]

2. Consider a single scalar field model of inflation, with an inflaton ϕ satisfying the field equation

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV(\phi)}{d\phi}$$

with potential $V(\phi) = \frac{1}{2}m^2\phi^2$.

(a) Explain what is meant by the *slow roll approximation* and why it is often a good approximation for making predictions from realistic models of inflation. [2]

(b) Find the equation of state $w \equiv P/\rho$ during slow-roll inflation in terms of the field value ϕ. [4]

(c) At what field value does inflation end? [1]

(d) After inflation ends the slow roll approximation is no longer valid and the scalar field oscillates about the minimum at $\phi = 0$. After a while H has decayed so that the damping term proportional to $3H\dot{\phi}$ can be neglected. Assuming there is negligible reheating and $3H\dot{\phi}$ can be neglected show that

$$\phi(t) = A\cos(mt + \alpha)$$

where α and A are constants. What is the condition on A for neglecting the damping term to be self-consistent? [3]

(e) Show that averaged over many oscillation times, the average pressure, $\bar{P}$, is zero. [3]

(f) Show that $\ddot{\phi} + \frac{dV}{d\phi} = 0$ can be written as

$$\dot{\phi}^2 = \phi\frac{dV}{d\phi} + \frac{d(\phi\dot{\phi})}{dt}.$$

Define an effective average equation of state during the oscillations $\bar{w} \equiv \bar{P}/\bar{\rho}$ where $\bar{\rho}$ is the density averaged over oscillations. Hence show now for a general power law potential $V(\phi) \propto \phi^n$ that the effective average equation of state late during the oscillations is

$$\bar{w} \approx \frac{n-2}{n+2}.$$

[Hint: note that $\frac{d(\phi\dot{\phi})}{dt}$ is a total derivative, so averaged over a long time it gives zero average for reasonable boundary conditions] [7]

3. During radiation domination the pressure is $P = \rho/3$ and $\delta P = \delta\rho/3$, so the sound speed is $c_s^2 = 1/3$. Neglecting any interactions the two linearised stress-energy conservation equations for scalar perturbations on sub-horizon scales are

$$\Delta' + \frac{4}{3}\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

$$\mathbf{v}' + \frac{1}{4}\boldsymbol{\nabla}\Delta + \boldsymbol{\nabla}\Psi = 0$$

where Δ is the radiation density perturbation and $\mathbf{v}$ is the velocity of the radiation fluid and other terms take their usual meanings.

(a) Derive an approximate solution for $\Delta(k, \eta)$ valid on small scales ($k \gg \mathcal{H}$) and explain briefly in your own words why the solution is oscillatory in η. [your solution can contain two unknown constants] [7]

(b) Derive an equation for $\mathbf{v}''$, and then use the Fourier transform definition

$$v \equiv \frac{1}{k}\frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{v})e^{-i\mathbf{k}\cdot\mathbf{x}}$$

to find an equation for v''. Derive the approximate general solution valid on small scales. [5]

(c) If on small scales $\Delta(k, \eta) = \Delta_0 \cos\left(k\eta/\sqrt{3}\right)$ find $v(k, \eta)$ in terms of Δ_0. [3]

(d) Approximating recombination at conformal time η_* as being radiation dominated, with the solution from part (c) for what wavenumbers is $\Delta(k, \eta_*)$ zero? At these wavenumbers what is the value of $v(k, \eta_*)$? [3]

(e) Explain briefly why the CMB power spectrum is never zero, even for rapid recombination where $\Delta(k, \eta_*)$ is exactly zero on specific scales. [2]