Early Universe : 2012 Open Note Test

Answer THREE questions out of four, each of which is marked out of 20.
Time allocated: 1Hr 30 minutes.

1. There are good reasons to think that the pressure $P \geq -\rho$, where ρ is the density. However consider the alternative, when the universe contains dark energy with a constant $w_{de} \equiv P_{de}/\rho_{de} < -1$. If a homogenous spatially-flat universe with scale factor a contains only this form of dark energy and pressureless matter with density ρ_m:

   (a) Show that the energy density of the dark energy increases with time. If the scale factor is $a_0 = 1$ today show that in the future

   $$\Omega_{de}(a) = \left(1 + \frac{\Omega_{m,0} a^{3w}}{\Omega_{de,0}}\right)^{-1}$$

   where $\Omega_{de,0}$ and $\Omega_{m,0}$ and the dark energy and matter densities as a fraction of the critical density today. If $\Omega_{de,0} = 0.75$ and $w = -2$, at roughly what scale factor is 99.9% of the energy density in dark energy? [6]

   (b) If the dark energy dominates the matter density at time t_{de} [so that $\rho_m(t_{de}) \ll \rho_{de}(t_{de})$], show that the universe has a "big rip" (scale factor $a \to \infty$) in a finite time Δt, and find Δt in terms of w_{de} and the Hubble parameter $H_{t_{de}}$ at time t_{de}. [7]

   (c) If the comoving Hubble parameter at time t_{de} is $\mathcal{H}_{t_{de}} \equiv H_{t_{de}} a_{t_{de}}$, what is the maximum comoving distance a photon can travel in the Δt from t_{de} until the big rip? Describe qualitatively how the angular scale and temperature of observed small-scale CMB anisotropies would change after t_{de} as the big rip is approached. [7]

2. The distribution function of photons in a homogeneous and isotropic photon gas in kinetic equilibrium is

   $$f_\gamma(E, T) = \frac{2}{(2\pi)^3} \frac{1}{e^{(E-\mu_\gamma)/T} - 1}.$$

   Consider the homogenous universe well after electron-positron annihilation is complete and ignore the very small effect of baryons.

   (a) At high temperatures $T \gg T_c \sim 0.5$keV double Compton scattering ($e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$) happens frequently in equilibrium. In this case explain why $\mu_\gamma = 0$. [3]

   (b) At lower temperatures $T \ll T_c$ double Compton scattering no longer happens, and in general μ_γ can be non-zero. As the gas cools below T_c it initially maintains its thermal distribution with $\mu_\gamma = 0$. If a small amount of energy is then injected into the photon gas to give an increase in the energy density by ϵ, show that after (rapid) kinetic thermalization

   $$\epsilon \approx \frac{T^4}{\pi^2} \int_0^\infty \frac{e^x x^3 dx}{(e^x - 1)^2} \left[\frac{\mu_\gamma}{T} + x\frac{\delta T}{T}\right]$$

   where the temperature is changed by δT and you can assume that $|\mu_\gamma/T| \ll 1$, $|\delta T/T| \ll 1$. [6]

   (c) If the energy injection increases the energy density *without* changing the number density of photons n_γ, show that after kinetic thermalization

   $$\frac{\mu_\gamma}{T} \approx \frac{\epsilon}{\rho_\gamma} \frac{C X_3}{X_3^2 - X_2 X_4}$$

   where X_k and C are defined to be the values of the integrals

   $$X_k \equiv \int_0^\infty \frac{x^k e^x dx}{(e^x - 1)^2} \qquad C \equiv \int_0^\infty \frac{x^3 dx}{e^x - 1}.$$

   [11]

3. Consider a single scalar field model of inflation, with potential $V(\phi)$. Assume slow roll inflation with parameters defined by

$$\epsilon_V \equiv \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta_V \equiv M_P^2 \frac{V_{,\phi\phi}}{V}$$

where M_P is the reduced Planck mass.

(a) Show that in terms of the Hubble parameter H one can write $\epsilon_V = -\frac{\dot{H}}{H^2}$. Explain why $\dot{H}$ is expected to be negative for most of the observable evolution of realistic inflation models. [7]

(b) A particular inflationary potential has an inflection point at field value ϕ_*, so that $V_{,\phi\phi} > 0$ for $\phi < \phi_*$ and $V_{,\phi\phi} < 0$ for $\phi > \phi_*$. Sketch an example of a potential with this property. If the field is rolling with $\dot{\phi} > 0$ and we observe perturbations that left the horizon when $\phi = \phi_*$, explain whether n_s (the spectral index of the power spectrum) is typically greater than or less than one, and in what case you could get $n_s = 1$. [4]

(c) The spectral index n_s does not have to be exactly the same at all scales, and sometimes a "running" parameter is defined

$$n_{\text{run}} \equiv \frac{d \ln n_s}{d \ln k}.$$

If we observe n_s at scales close to where modes left the horizon at η_*, explain what sign of n_{run} we would expect to observe if we also have $V_{,\phi} = 0$ at $\phi = \phi_*$ (the inflection point). [5]

(d) Assume that the values of A_s and n_s are known at wavenumber $k_* = 0.01\text{Mpc}^{-1}$ corresponding roughly to the scale of the turnover in the matter power spectrum. Describe qualitatively (perhaps with two sketches) how you would expect the shape of the matter power spectrum to be different for $n_{\text{run}} < 0$ and $n_{\text{run}} > 0$ if all other cosmological parameters remain the same. [4]

4. Neglecting any small effect of pressure, a sub-horizon matter density ρ_i (e.g. i=b, $i = c$ or $i = m$ for baryons, dark matter or the total density respectively) has a fractional linear perturbation Δ_i evolving with

$$\Delta_i'' + \mathcal{H}\Delta_i' = \boldsymbol{\nabla}^2\Psi$$

where a dash denotes a derivative with respect to conformal time η. Consider a flat linearly-perturbed FRW universe in the epoch of matter domination, with a density ρ_b of baryons and ρ_c of dark matter, and neglect all pressures.

(a) Show from the Friedmann equation that the comoving Hubble parameter during matter domination is given by $\mathcal{H} = 2/\eta$. [4]

(b) Show that there is a solution $\Delta_m \propto \eta^2$ where Δ_m is the fractional perturbation in the total matter energy density. [5]

(c) Derive an equation for the evolution of $\Delta \equiv \Delta_c - \Delta_b$. where Δ_c and Δ_b are the fractional perturbations in the dark matter and baryon densities.

At the end of recombination at time η_* (approximated to be matter dominated), on small scales the baryon density is very smooth because of Silk damping so that $\Delta_b(\eta_*) = \Delta_b'(\eta_*) = 0$. Assuming that $\rho_b \ll \rho_c$, so that $\Delta_c \sim \Delta_m$, show that after recombination on these scales

$$\frac{\Delta_b}{\Delta_c} \approx 1 - 3\left(\frac{\eta_*}{\eta}\right)^2 + 2\left(\frac{\eta_*}{\eta}\right)^3.$$

Explain briefly the physics of what is happening to the matter perturbations after recombination and why at late times ($\eta \gg \eta_*$) we have $\Delta_b \approx \Delta_c$. [11]