Cosmology: Problem Sheet 2

Deadline: 6th Nov (Week 8), Wednesday 12:00 (School Office hand in)

1 Bose-Einstein distribution

1. Explain in your own words what is meant by thermal equilibrium. [6]

2. For bosons, the number of ways of arranging $N = \sum n_i$ particles in quantum states with energies $\epsilon_i$ and $g_i$ substates at each energy is

$$W \equiv \prod_i w_b(n_i, g_i) = \prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}.$$ 

Use this to derive the maximum entropy Bose-Einstein distribution for the average occupation numbers of each quantum state $n_i = \frac{1}{e^{\alpha + \beta \epsilon_i} - 1}$, where $\alpha$ and $\beta$ are constants that label the macrostate with fixed number of particles and energy. You can assume that $n_i \gg 1$, $g_i \gg 1$. [9]

3. The maximum entropy state has $\hat{n}_i$ particles with energy $\epsilon_i$. Consider Taylor expanding $\ln W$ about its maximal value, so

$$\ln W \approx \ln \hat{W} + \frac{1}{2} \sum_i (n_i - \hat{n}_i)^2 \frac{\partial^2 \ln W}{\partial n_i^2} \bigg|_{n_i = \hat{n}_i},$$

where $\hat{W}$ is $W$ evaluated for the maximum entropy value values $n_i = \hat{n}_i$. With this approximation show that

$$W \approx \hat{W} \exp \left(- \sum_i \frac{(n_i - \hat{n}_i)^2}{2\sigma_i^2} \right)$$

where $\sigma_i^2 = \hat{n}_i(1 + N_i)$. For large $n_i$ what is the fractional standard deviation of each $n_i$ from the maximum entropy value (in terms of $\hat{n}_i$), and what is the physical significance of this result? [7,3]

2 Equilibrium densities

Here we do some bookwork to verify the results in the notes. The equilibrium distribution function for a species A is given by

$$f_A(p,t) = \frac{g_A}{(2\pi)^3} \frac{1}{\exp[(E_A - \mu_A)/T_A] \pm 1}$$

where + sign is for fermions - sign is for bosons and $E_A(p) = \sqrt{p^2 + m_A^2}$. $g_A$ is the spin degeneracy factor, $\mu_A$ is the chemical potential and $T_A$ the temperature of the species A.

Derive the following form of the number density $n$, density $\rho$ and pressure $P$ for relativistic species ($m \ll T$) with negligible chemical potential ($\mu \ll T$).

$$n_B = \frac{g \zeta(3)}{\pi^\frac{3}{2}} T^3, \quad \rho_B = \frac{g \pi^2}{30} T^4, \quad P_B = \frac{1}{3} \rho_B$$

$$n_F = \frac{3}{4} n_B, \quad \rho_F = \frac{7}{8} \rho_B, \quad P_F = \frac{1}{3} \rho_F$$

where $B$=bosons and $F$=fermions. [20]
3 Extra relativistic species

According to the standard assumptions, there are three species of (massless) neutrinos. In the temperature range of \(1 \text{MeV} < T < 100 \text{MeV}\), the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos all of which were in thermal equilibrium.

1. Neglecting any change in the degrees of freedom at \(T > 100 \text{MeV}\), show using the Friedmann equation for a flat radiation-dominated universe \(H^2 = 8\pi G \rho / 3\) that for temperatures \(T > 1 \text{MeV}\) the time since the start of the hot big bang is given by

\[
t(T) = \left( \frac{A}{g_*} \right)^{1/2} \frac{M_P}{T^2}
\]

where \(M_P \equiv \sqrt{\hbar c / (8\pi G)}\) is the reduced Planck mass, and \(A\) is a constant that you should give explicitly. What is \(g_*\)? Put in \(\hbar, c\) and \(k_B\) factors give a result in standard rather than natural units.

How long did it take from the big bang for the temperature to fall to \(T = 1 \text{MeV}\)? \([\text{Give the result in seconds}].\) \([10,4]\)

2. How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming? \([4]\)

3. What would be the mass density (in \(\text{kg/m}^3\)) of the universe when \(T = 1 \text{MeV}\) under the standard assumptions, and what would it be if there were one other species of massless neutrino? What is the temperature in Kelvin at \(T = 1 \text{MeV}\), and what is the redshift? \([6,4]\)

4. What approximation have you made about the electrons and positron velocities, and is it reasonable? \([2]\)

4 CMB blackbody and \(\mu\)-distortions

The distribution function of photons in a homogeneous and isotropic photon gas in kinetic equilibrium is

\[
f_\gamma(E, T) = \frac{2}{(2\pi)^3} \frac{1}{e^{(E - \mu_\gamma)/T} - 1}.
\]

Consider the homogenous universe well after electron-positron annihilation is complete and ignore the very small effect of baryons.

1. At high temperatures \(T \gg T_c \sim 0.5 \text{keV}\) double Compton scattering \((e^- + \gamma \leftrightarrow e^- + \gamma + \gamma)\) happens frequently in equilibrium. In this case explain why \(\mu_\gamma = 0\). \([4]\)

2. At lower temperatures \(T \ll T_c\) double Compton scattering no longer happens, and in general \(\mu_\gamma\) can be non-zero. As the gas cools below \(T_c\) it initially maintains its thermal distribution with \(\mu_\gamma = 0\). If a small amount of energy is then injected into the photon gas to give an increase in the energy density by \(\epsilon\), show by doing a first order series expansion in \(\delta T\) and \(\mu_\gamma\) that after (rapid) kinetic thermalization

\[
\epsilon \approx \frac{T^4}{\pi^2} \int_0^\infty e^{x^2} dx \frac{x^3}{(e^x - 1)^2} \left( \frac{\mu_\gamma}{T} + x \frac{\delta T}{T} \right),
\]

where the temperature is changed by \(\delta T\) and the chemical potential is changed by \(\mu_\gamma\) (from zero); you can assume that \(|\mu_\gamma/T| \ll 1, |\delta T/T| \ll 1\). \([10]\)

3. If the energy injection increases the energy density without changing the number density of photons \(n_\gamma\), show that after kinetic thermalization

\[
\frac{\mu_\gamma}{T} \approx \frac{\epsilon}{n_\gamma} \frac{C X_3}{X_2^3 - X_1 X_4}
\]

where \(X_k\) and \(C\) are defined to be the values of the integrals

\[
X_k = \int_0^\infty \frac{x^k e^{x^2}}{(e^x - 1)^2} \quad C = \int_0^\infty \frac{x^3 e^x}{e^x - 1}.
\]

[This shows that processes depositing energy at \(T < 0.5 \text{keV}\) can give rise to a “\(\mu\)-distortion” in the CMB, i.e. a not-exactly blackbody spectrum.] \([11]\)