Cosmology: Problem Sheet 2

Deadline: 6th Nov (Week 8), Wednesday 12:00 (School Office hand in)

1 Bose-Einstein distribution

1. Explain in your own words what is meant by *thermal equilibrium*. [6]

2. For bosons, the number of ways of arranging $N = \sum n_i$ particles in quantum states with energies ϵ_i and g_i substates at each energy is

$$W \equiv \prod_i w_b(n_i, g_i) = \prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}.$$

Use this to derive the maximum entropy Bose-Einstein distribution for the average occupation numbers of each quantum state

$$\mathcal{N}_i = \frac{1}{e^{\alpha+\beta\epsilon_i} - 1},$$

where α and β are constants that label the macrostate with fixed number of particles and energy. You can assume that $n_i \gg 1$, $g_i \gg 1$. [9]

3. The maximum entropy state has $\hat{n}_i$ particles with energy ϵ_i. Consider Taylor expanding $\ln W$ about its maximal value, so

$$\ln W \approx \ln \hat{W} + \frac{1}{2}\sum_i (n_i - \hat{n}_i)^2 \left.\frac{\partial^2 \ln W}{\partial n_i^2}\right|_{n_i=\hat{n}_i},$$

where $\hat{W}$ is W evaluated for the maximum entropy value values $n_i = \hat{n}_i$. With this approximation show that

$$W \approx \hat{W} \exp\left(-\sum_i \frac{(n_i - \hat{n}_i)^2}{2\sigma_i^2}\right)$$

where $\sigma_i^2 = \hat{n}_i(1 + \mathcal{N}_i)$. For large n_i what is the fractional standard deviation of each n_i from the maximum entropy value (in terms of $\hat{n}_i$), and what is the physical significance of this result? [7,3]

2 Equilibrium densities

Here we do some bookwork to verify the results in the notes. The equilibrium distribution function for a species A is given by

$$f_A(p,t) = \frac{g_A}{(2\pi)^3} \frac{1}{\exp[(E_A - \mu_A)/T_A] \pm 1} \tag{1}$$

where + sign is for fermions - sign is for bosons and $E_A(p) = \sqrt{p^2 + m_A^2}$. g_A is the spin degeneracy factor, μ_A is the chemical potential and T_A the temperature of the species A.

Derive the following form of the number density n, density ρ and pressure P for relativistic species ($m \ll T$) with negligible chemical potential ($\mu \ll T$).

$$\begin{aligned} n_B &= \frac{g\,\zeta(3)}{\pi^2} T^3\,, \qquad \rho_B = \frac{g\,\pi^2}{30} T^4\,, \qquad P_B = \frac{1}{3}\rho_B \\ n_F &= \frac{3}{4} n_B\,, \qquad \rho_F = \frac{7}{8}\rho_B\,, \qquad P_F = \frac{1}{3}\rho_F \end{aligned} \tag{2}$$

where B=bosons and F=fermions. [20]

3 Extra relativistic species

According to the standard assumptions, there are three species of (massless) neutrinos. In the temperature range of $1MeV < T < 100MeV$, the mass density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and these neutrinos all of which were in thermal equilibrium.

1. Neglecting any change in the degrees of freedom at $T > 100MeV$, show using the Friedmann equation for a flat radiation-dominated universe $H^2 = 8\pi G\rho_R/3$ that for temperatures $T > 1MeV$ the time since the start of the hot big bang is given by

$$t(T) = \left(\frac{A}{g_*}\right)^{1/2} \frac{M_P}{T^2}$$

   where $M_P \equiv \sqrt{\hbar c/(8\pi G)}$ is the reduced Planck mass, and A is a constant that you should give explicitly. What is g_*? Put in $\hbar$, c and k_B factors give a result in standard rather than natural units. How long did it take from the big bang for the temperature to fall to $T = 1MeV$? [Give the result in seconds]. [10,4]

2. How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming? [4]

3. What would be the mass density (in kg/m^3) of the universe when $T = 1MeV$ under the standard assumptions, and what would it be if there were one other species of massless neutrino? What is the temperature in Kelvin at $T = 1MeV$, and what is the redshift? [6,4]

4. What approximation have you made about the electrons and positron velocities, and is it reasonable? [2]

4 CMB blackbody and μ-distortions

The distribution function of photons in a homogeneous and isotropic photon gas in kinetic equilibrium is

$$f_\gamma(E,T) = \frac{2}{(2\pi)^3} \frac{1}{e^{(E-\mu_\gamma)/T} - 1}.$$

Consider the homogenous universe well after electron-positron annihilation is complete and ignore the very small effect of baryons.

1. At high temperatures $T \gg T_c \sim 0.5\text{keV}$ double Compton scattering ($e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$) happens frequently in equilibrium. In this case explain why $\mu_\gamma = 0$. [4]

2. At lower temperatures $T \ll T_c$ double Compton scattering no longer happens, and in general μ_γ can be non-zero. As the gas cools below T_c it initially maintains its thermal distribution with $\mu_\gamma = 0$. If a small amount of energy is then injected into the photon gas to give an increase in the energy density by ϵ, show by doing a first order series expansion in δT and μ_γ that after (rapid) kinetic thermalization

$$\epsilon \approx \frac{T^4}{\pi^2} \int_0^\infty \frac{e^x x^3 dx}{(e^x - 1)^2} \left[\frac{\mu_\gamma}{T} + x\frac{\delta T}{T}\right],$$

   where the temperature is changed by δT and the chemical potential is changed by μ_γ (from zero); you can assume that $|\mu_\gamma/T| \ll 1$, $|\delta T/T| \ll 1$. [10]

3. If the energy injection increases the energy density *without* changing the number density of photons n_γ, show that after kinetic thermalization

$$\frac{\mu_\gamma}{T} \approx \frac{\epsilon}{\rho_\gamma} \frac{CX_3}{X_3^2 - X_2 X_4}$$

   where X_k and C are defined to be the values of the integrals

$$X_k \equiv \int_0^\infty \frac{x^k e^x dx}{(e^x - 1)^2} \qquad C \equiv \int_0^\infty \frac{x^3 dx}{e^x - 1}.$$

   [This shows that processes depositing energy at $T < 0.5\text{keV}$ can give rise to a "μ-distortion" in the CMB, i.e. a not-exactly blackbody spectrum.] [11]