Cosmology : Answer Sheet 0

I. HUBBLE PARAMETER

The Hubble parameter is estimated from Hubble’s law using \( H_0 = v/r = zc/r \). If there is an error \( \delta v \) on \( v \) then \( (\delta H_0)/H_0 = \delta v/v = \delta v/(H_0r) \). Hence for 10% error \( \delta v = 0.1H_0r \) so need \( r \geq \delta v/(0.1H_0) \).

If the rms velocity is 600 km s\(^{-1}\) (which is in random directions), the rms velocity along the line of sight (which is what is measured by the redshift) is \( \delta v = \sqrt{600^2/3} \text{ km/s} \approx 3.46 \times 10^5 \text{ m/s} \). Hence \( r \geq 3.46 \times 10^5/0.1/(h \times 10^5) \text{ Mpc} \approx 35h^{-1} \text{ Mpc} \). i.e. if \( h = 1 \) then \( r \geq 35 \text{ Mpc} \), if \( h = 0.7 \) then \( r \geq 50 \text{ Mpc} \).

II. NEWTONIAN FRIEDMANN EQUATIONS WITH COSMOLOGICAL CONSTANT

Since \( F/m = -\partial/\partial_r V \) we have

\[
V = -\frac{GM}{r} + \frac{\Lambda r^2}{6} + \text{const} = -\frac{4\pi G\rho r^2}{3} - \frac{\Lambda r^2}{6} + \text{const}
\]

The points should be moving apart, with kinetic energy in a spherical shell of mass \( m \) (on any particle in it) is given by

\[
T = \frac{mv^2}{2} = \frac{m}{a^2} \dot{a}^2 / 2.
\]

The spherical shell has potential energy \( V(r)m \) and so the total energy is (up to the constant)

\[
U = T + V = \left( \frac{\dot{a}^2}{2} - \frac{4\pi G\rho r^2}{3} - \frac{\Lambda r^2}{6} \right) m
\]

This energy should be conserved as the particles move apart. Let’s take out the expansion using \( r = a\chi \) giving

\[
\frac{U}{ma^2\chi^2} = \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 - \frac{4\pi G\rho}{3} - \frac{\Lambda}{6} \chi^{-2}
\]

The RHS is not a function of \( \chi \) (homogeneity assumption), so the LHS cannot be either, and we can define a constant \( K = -2U/(ma^2) \) so that

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G\rho \frac{3}{3} - \frac{K}{a^2} + \frac{\Lambda}{3}.
\]

Taking the time derivative

\[
2 \left( \frac{\dot{a}}{a} \right) \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) = 8\pi G\rho \frac{3}{3} + 2K\dot{a} = -8\pi GH\rho + \frac{2KH}{a^2}.
\]

where we used the energy conservation equation with \( P = 0 \) and substituted \( \dot{a} \) for \( H \). Using the Friedmann eq on the LHS then gives

\[
2H \left( \frac{\ddot{a}}{a} - \frac{8\pi G\rho}{3} - \frac{K}{a^2} - \frac{\Lambda}{3} \right) = -8\pi GH\rho + \frac{2KH}{a^2}
\]

\[
\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3}
\]
III. BIG RIP

1. The energy conservation equation is

\[ \dot{\rho}_{\text{de}} + 3H(\rho_{\text{de}} + P_{\text{de}}) = 0. \]

For \( P_{\text{de}} = w \rho_{\text{de}} \) we have

\[ \frac{\dot{\rho}_{\text{de}}}{\rho_{\text{de}}} = -3(1 + w) \frac{\dot{a}}{a}. \]

Hence

\[ d(\ln \rho_{\text{de}}) = -3(1 + w)d\ln a \implies \rho_{\text{de}} \propto a^{-3(1+w)}. \]

For \( w < -1 \) we have \( 1 + w < 0 \) and hence \( \rho_{\text{de}} = \rho_{\text{de},0}a^{3(1+w)} \) increases with time. Hence

\[ \Omega_{\text{de}}(a) = \frac{\rho_{\text{de}}}{\rho_{\text{de}} + \rho_{m}} \bigg|_{a} = \frac{\rho_{\text{de},0}a^{-3(1+w)}}{\rho_{\text{de},0}a^{-3(1+w)} + \rho_{m,0}a^{-3}} = \frac{\Omega_{\text{de},0}}{\Omega_{\text{de},0} + \Omega_{m,0}a^{3w}} \]

which diving through by \( \Omega_{\text{de},0} \) gives the answer. For \( \Omega_m(a) \) small,

\[ \Omega_{\text{de}}(a) \approx 1 - \frac{\Omega_{m,0}a^{3w}}{\Omega_{\text{de},0}}, \]

hence for \( \Omega_{\text{de}}(a) = 0.999 \)

\[ \frac{\Omega_{m,0}a^{3w}}{\Omega_{\text{de},0}} \approx 0.001 \implies a \approx (0.003)^{-1/6} \approx 2.63. \]

2. For \( \rho_{\text{de}} \propto a^{-3(1+w)} \), with \( \rho \approx \rho_{\text{de}} \) so that \( \Omega_{\text{de}} \approx 1 \)

\[ H = H_{t_{\text{de}}} \left( \frac{a}{t_{\text{de}}} \right) ^{-3(1+w)/2} = \frac{1}{a} \frac{da}{dt} \]

Hence

\[ \int_{a_{\text{de}}}^{a} a^{(1+3w)/2} da = a_{\text{de}}^{3(1+w)/2} H_{t_{\text{de}}} \int_{t_{\text{de}}}^{t_{\text{de}} + \Delta t} dt \]

and for \( w < -1 \) the integral converges for \( a' \to \infty \) giving

\[ \Delta t = H_{t_{\text{de}}}^{-1} a_{\text{de}}^{-3(1+w)/2} \left[ \frac{2}{3(1+w)} a_{\text{de}}^{3(1+w)/2} \right]_{a_{\text{de}}}^{\infty} = \frac{2H_{t_{\text{de}}}^{-1}}{3(1+w)} \]

3. Since \( a \to \infty \), photons become increasingly redshifted (wavelength becomes increasingly long, so photons have increasingly low energy and so become unobservable).