Cosmology : Answer Sheet 0

I. HUBBLE PARAMETER

The Hubble parameter is estimated from Hubble's law using $H_0 = v/r = zc/r$. If there is an error δv on v then $(\delta H_0)/H_0 = \delta v/v = \delta v/(H_0 r)$. Hence for 10% error $\delta v = 0.1 H_0 r$ so need $r \geq \delta v/(0.1 H_0)$.

If the rms velocity is $600\,\mathrm{km\,s^{-1}}$ (which is in random directions), the rms velocity along the line of sight (which is what is measured by the redshift) is $\delta v = \sqrt{600^2/3}$ km/s $\approx 3.46 \times 10^5$m/s. Hence $r \geq 3.46 \times 10^5/0.1/(h \times 10^5) Mpc \approx 35h^{-1}$Mpc. i.e. if $h = 1$ then $r \geq 35$Mpc, if $h = 0.7$ then $r \geq 50$Mpc.

II. NEWTONIAN FRIEDMANN EQUATIONS WITH COSMOLOGICAL CONSTANT

Since $F/m = -\partial/\partial_r V$ we have

$$V = -\frac{GM}{r} + \frac{\Lambda r^2}{6} + \mathrm{const} = -\frac{4\pi G\rho r^2}{3} - \frac{\Lambda r^2}{6} + \mathrm{const}$$

The points should be moving apart, with kinetic energy in a spherical shell of mass m (on any particle in it) is given by

$$T = mv^2/2 = m\dot{r}^2/2.$$

The spherical shell has potential energy $V(r)m$ and so the total energy is (up to the constant)

$$U = T + V = \left(\frac{\dot{r}^2}{2} - \frac{4\pi G\rho r^2}{3} - \frac{\Lambda r^2}{6}\right) m$$

This energy should be conserved as the particles move apart. Let's take out the expansion using $r = a\chi$ giving

$$\frac{U}{ma^2\chi^2} = \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{4\pi G\rho}{3} - \frac{\Lambda}{6}.$$

The RHS is not a function of χ (homogeneity assumption), so the LHS cannot be either, and we can define a constant $K = -2U/(m\chi^2)$ so that

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2} + \frac{\Lambda}{3}.$$

Taking the time derivative

$$\begin{aligned} 2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) &= \frac{8\pi G\dot{\rho}}{3} + \frac{2K\dot{a}}{a^3} \\ &= -8\pi GH\rho + \frac{2KH}{a^2}. \end{aligned}$$

where we used the energy conservation equation with $P = 0$ and substituted $\dot{a}$ for H. Using the Friedmann eq on the LHS then gives

$$2H\left(\frac{\ddot{a}}{a} - \frac{8\pi G\rho}{3} + \frac{K}{a^2} - \frac{\Lambda}{3}\right) = -8\pi GH\rho + \frac{2KH}{a^2}$$

$$\implies \frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3}$$

III. BIG RIP

1. The energy conservation equation is

$$\dot{\rho}_{de} + 3H(\rho_{de} + P_{de}) = 0.$$

For $P_{de} = w\rho_{de}$ we have

$$\frac{\dot{\rho_{de}}}{\rho_{de}} = -3(1+w)\frac{\dot{a}}{a}$$

Hence

$$d(\ln \rho_{de}) = -3(1+w)d\ln a \implies \rho_{de} \propto a^{-3(1+w)}.$$

For $w < -1$ we have $1 + w < 0$ and hence $\rho_{de} = \rho^0_{de} a^{3|1+w|}$ increases with time. Hence

$$\Omega_{de}(a) = \left.\frac{\rho_{de}}{\rho_{de} + \rho_m}\right|_a = \frac{\rho_{de,0}a^{-3(1+w)}}{\rho_{de,0}a^{-3(1+w)} + \rho_{m,0}a^{-3}} = \frac{\Omega_{de,0}}{\Omega_{de,0} + \Omega_{m,0}a^{3w}}$$

which diving through by $\Omega_{de,0}$ gives the answer. For $\Omega_m(a)$ small,

$$\Omega_{de}(a) \approx 1 - \frac{\Omega_{m,0}a^{3w}}{\Omega_{de,0}},$$

hence for $\Omega_{de}(a) = 0.999$

$$\frac{\Omega_{m,0}a^{3w}}{\Omega_{de,0}} \approx 0.001 \implies a \approx (0.003)^{-1/6} \approx 2.63.$$

2. For $\rho_{de} \propto a^{-3(1+w)}$, with $\rho \approx \rho_{de}$ so that $\Omega_{de} \approx 1$

$$H = H_{t_{de}} \left(\frac{a}{a_{t_{de}}}\right)^{-3(1+w)/2} = \frac{1}{a}\frac{da}{dt}$$

Hence

$$\int_{a_{de}}^{a'} a^{(1+3w)/2} da = a_{t_{de}}^{3(1+w)/2} H_{t_{de}} \int_{t_{de}}^{t_{de}+\Delta t} dt$$

and for $w < -1$ the integral converges for $a' \to \infty$ giving

$$\Delta t = H_{t_{de}}^{-1} a_{t_{de}}^{-3(1+w)/2} \left[\frac{2}{3(1+w)} a^{3(1+w)/2}\right]_{a_{t_{de}}}^{\infty} = \frac{2H_{t_{de}}^{-1}}{3|1+w|}.$$

3. Since $a \to \infty$, photons become increasingly redshifted (wavelength becomes increasingly long, so photons have increasingly low energy and so become unobservable).