Plane symmetric systems

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1 Introduction

Few problems can be solved analytically either in GTG or in GR, but it is interesting to study the few cases where the solutions are tractable. Here we look at systems with high symmetry — plane symmetric systems. An example of such a system that may be relevant to cosmology is the domain wall, though of course the plane symmetry will only be an approximation in any realistic case. Plane symmetric systems are also interesting because of their various pathalogical properties, as we shall show. In the case of a black hole the GTG approach has aided understanding by motivating different choices of gauge. We shall see if we can do something similar here, or if the pathologies really are intrinsic to plane symmetric systems.

2 Einstein's Equations

We start with the following ansätz consistent with plane symmetry

$$\begin{aligned} g^t &= f_1 e^t + f_2 e^x \\ g^x &= g_1 e^x + g_2 e^t \\ g^y &= \alpha e^y \\ g^z &= \alpha e^z \end{aligned}$$

where f_1, f_2, g_1, g_2, α are functions only of x and t.

We can perform rotation and position gauge transformations on this. We use the rotation gauge freedom to diagonalize the stress-energy tensor. An x and t dependant rescaling along the e_t direction can be used to set $f_2 = 0$. Similarly we can rescale along the e_x direction to set $g_2 = 0$. Alternatively we could set $g_1 = 1$ and keep a g_2 which may have advantages since it makes x the proper distance co-ordinate so results would be easier to interpret. In cases where we do not need to use the rotation gauge freedom to set the covariant fluid velocity to e_t we have the extra freedom to set $g_1 = 1$ and $g_2 = 0$. For the moment we just set $f_2 = 0$.

In the absence of torsion we can calculate the covariant rotation gauge function ω_a from the $\overline{h}$-function using[?]

$$\omega_a = -H_a + \tfrac{1}{2} a\cdot(b\wedge H^b)$$

where

$$H_a = \overline{h}(\nabla\wedge\overline{h}_a^{-1}).$$

The result is

$$\begin{aligned} \omega_t &= A e_x e_t \\ \omega_x &= B e_x e_t \\ \omega_y &= F e_y e_t + G e_y e_x \\ \omega_z &= F e_z e_t + G e_z e_x \end{aligned}$$

where

$$\begin{aligned}
A &= -\frac{L_x f_1}{f_1} && (1)\\
B &= -\frac{L_t g_1}{g_1} + \frac{L_x g_2}{g_1} - g_2 \frac{L_x f_1}{g_1 f_1} && (2)\\
F &= -\frac{L_t \alpha}{\alpha} && (3)\\
G &= \frac{L_x \alpha}{\alpha} && (4)
\end{aligned}$$

Since we have fixed the rotation gauge so that the stress-energy tensor is diagonal we can define the diagonal elements ρ, P_x and P_{yz}. Using units where $8\pi G_N = c = 1$ Einstein's equation then gives us the set of equations

$$2L_x G = 3G^2 - F^2 - 2BF + \rho \tag{5}$$
$$2L_t F = G^2 - 3F^2 - 2AG - P_x \tag{6}$$
$$L_t B - L_x A = F^2 - G^2 + A^2 - B^2 + \tfrac{1}{2}(P_x - 2P_{yz} - \rho) \tag{7}$$
$$L_t G = -F(G + A) \tag{8}$$
$$L_x F = G(F - B) \tag{9}$$

The Bianchi identities give two further equations

$$L_t \rho = -B(\rho + P_x) - 2F(\rho + P_{yz}) \tag{10}$$
$$L_x P_x = -A(\rho + P_x) + 2G(P_x - P_{yz}) \tag{11}$$

The equations are invariant under

$$\begin{array}{cccc}
A \to B & B \to -A & F \to G & G \to -F \\
L_x \to L_t & L_t \to -L_x & P_{yz} \to -P_{yz} & \rho \leftrightarrow P_x
\end{array}$$

The Weyl tensor is

$$\mathcal{W}(B) = \frac{1}{4}(B + 3\sigma_x B \sigma_x)(G^2 - F^2 - \frac{1}{3}[P_x + P_{yz} + \rho]).$$

Assuming F and G are non-zero we can re-write two of the Einstein equations as

$$\begin{aligned}
\rho\, G &= (L_x - 3G)(G^2 - F^2) \\
P_x\, F &= (L_t + 3F)(G^2 - F^2) && (12)
\end{aligned}$$

The integrating factor for these equations is α^{-3}, though there seems no simple use for this fact.

As a quick check we can recover the flat Friedmann equations by setting $\alpha = g_1 = \alpha(t)$ and $P_x = P_{yz} = P(t)$ which implies that $G = 0$ and $F = B$. The Bianchi identity (11) then gives $A = 0$ and hence we can set $f_1 = 1$. Equation (5) now yields $\rho = 3F^2$ and (7) gives $\partial_t F = -\frac{1}{2}(\rho + P)$. These are the Friedmann equations where F is the Hubble parameter. The evolution of ρ is given by the first Bianchi identity (10) and is $\partial_t \rho = -3F(\rho + P)$. The standard form is recovered by writing $S(t) = \alpha(t)^{-1}$ so that $F = \partial_t S/S$.

3 The time-independant case

Setting $f_2 = g_2 = 0$ we have

$$G = \frac{g_1 \partial_x \alpha}{\alpha} \qquad F = B = 0 \qquad A = -\frac{g_1 \partial_x f_1}{f_1}$$

and equation (5) becomes

$$2L_x G = 3G^2 + \rho.$$

Since for reflection symmetric systems we require $\partial_x \alpha = 0$ at $x = 0$, G is zero at the origin and will therefore become singular at a finite distance if the energy density is positive and g_1 is positive. Since we want $g_1 > 0$ and $F = 0$ this implies a singularity in $\mathcal{W}(B)$. So there are no time independant non-singular plane symmetric solutions.

4 A thin wall

If we assume that the wall is negligibly thin (6) tells us that the integrated P_x is zero, so P_x is zero everywhere. Substituting in (6) we immediately get $2A = G$ (since $G = 0$ is uninteresting) and using the form of A and G we have

$$\alpha = c/f_1^2$$

where c is a constant of integration. We can perform a constant rescaling to set $c = 1$.

If we also set $\rho = 0$, (5) becomes

$$2L_x G = 3G^2$$

choosing a gauge where $g_1 = f_1$ we have

$$2g_1 \partial_x G = 3G^2 \quad \text{and} \quad G = -2\partial_x g_1$$

This implies that

$$\partial_x (g_1^3 \partial_x g_1) = 0$$

so there is a solution

$$g_1 = (cx + d)^{\frac{1}{4}}.$$

We can rescale to set $d = 1$. This solution will be valid for $x > 0$ only since we require the solution to be reflection symmetric. We therefore want

$$g_1 = (1 + k|x|)^{\frac{1}{4}}.$$

From (5) we know that the energy density of the wall $\sigma = 2\Delta G$, where ΔG is the change in G across $x = 0$. Here

$$G = -\tfrac{1}{2}(1 + k|x|)^{-3/4} k \partial_x |x|$$

so $\sigma = -2k$. Therefore

$$g_1 = (1 - \sigma/2|x|)^{\frac{1}{4}}$$

This corresponds to the metric usually quoted for a time independant thin wall solution[1]

$$ds^2 = (1 - \sigma/2|x|)^{-\frac{1}{2}}(dt^2 - dx^2) - (1 - \sigma/2|x|)(dy^2 + dz^2).$$

It's no good for a domain wall since $P_{yz} = -\rho/4$. It's interesting that the gauge fields only cover part of space—for $|x| > 2/\sigma$ there is no solution for g_1. Unfortunately it is only a finite proper distance to this point from the wall. Here we chose $f_1 = g_1$ in order to recover the solution as it is usually seen. Choosing the $g_1 = 1$ gauge naturally leads to the same physical conclusions.

5 The time dependant thin walls

With $g_2 = f_2 = 0$ we have

$$G = \frac{g_1 \partial_x \alpha}{\alpha} \qquad F = -\frac{f_1 \partial_t \alpha}{\alpha} \qquad A = -\frac{g_1 \partial_x f_1}{f_1} \qquad B = -\frac{f_1 \partial_t g_1}{g_1}.$$

In general we have no additional freedom in the choice of g_1 and f_1, however for vacuum solutions we can use the extra rotation gauge freedom to set $g_1 = 1$, so that $B = 0$. If we look for solutions in which α is separable we also have $\partial_t G = 0$. Equation (8) then implies that $A = -G$. Now summing equations (5) and (7) we get

$$L_x G = G^2 \tag{13}$$

and the solution is

$$G = (c - x)^{-1}.$$

If the mass of the wall is positive c will also be positive so we see that G will always become singular at a finite distance from the wall. A singularity in G does not neccessarily mean the presense of a curvature singularity, for instance in empty space if $F^2 = G^2$ the Weyl tensor will be zero. It's not clear whether making α separable is in general valid.

The corresonding solution for α is

$$\alpha = (c - x)^{-1} \nu(t).$$

Performing constant rescalings to rearrange the constants, choosing $\partial_t f_1 = 0$, using (9) to solve for f_1 and imposing reflection symmetry we have

$$f_1 = (1 - k|x|)^{-1} \qquad \alpha = f_1 \nu(t).$$

Equations (5) and (7) now show that $F^2 = G^2$ so the solution has $\mathcal{W}(B) = 0$. We can now solve for $\nu(t)$ to get

$$\alpha = f_1 e^{\pm kt}.$$

corresponding to the metric

$$ds^2 = (1 - k|x|)^2 dt^2 - dx^2 - (1 - k|x|)^2 e^{\mp 2kt}(dy^2 + dz^2).$$

As before the wall mass is $\sigma = 2\Delta G = 4k$, so that $k = \sigma/4$. We can calculate the integrated P_x and P_{yz} from equations (6) and (7), so that $P_x = 0$ and $\rho = -P_{yz}$, just what we want for a thin domain wall, but ruling out thin walls of dust. The sign ambiguity in $F = \pm G$ just corresponds to time reversal (though note that (9) shows that the sign of FG changes across $x = 0$). By considering the geodesic equations it's easy to show that the wall is repulsive, geodesic observers accellerate away from it. This is all very well known.[3] In GR one can transform to Minkowski co-ordinates of the geodesic observer, in which frame the wall apparently becomes a sphere! I'm not sure how to reproduce anything similar, especially given that it invloves an analytic continuation.

6 Domain walls

Consider the prototype Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\mathcal{D}\phi)^2 - V$$

for a real scalar field ϕ with a symmetry breaking double-well potential, for instance

$$V = V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2.$$

The corresponding action is

$$S = \int |d^4x| \det(\underline{h})^{-1}\mathcal{L}.$$

yielding the field equation

$$\mathcal{D}\cdot\mathcal{D}\phi - \partial_\phi V = 0$$

and stress-energy tensor

$$\mathcal{T}(a) = a\cdot\mathcal{D}\phi\mathcal{D}\phi - a\mathcal{L}.$$

We shall look for time independant solutions. In the non-gravitationally coupled case $\mathcal{T}(e_x) = 0$ because the field equation can be integrated to give

$$\tfrac{1}{2}(\partial_x\phi)^2 - V = 0.$$

Here the field equation is

$$L_x L_x\phi + (A - 2G)L_x\phi - \partial_\phi V = 0$$

and the stess-energy tensor is

$$\mathcal{T}(a) = \tfrac{1}{2}e_x a e_x (L_x\phi)^2 + aV$$

so that $\rho = -P_{yz}$. However we can only write

$$\tfrac{1}{2}(L_x\phi)^2 - V = 0$$

if $A = 2G$. So P_x is not zero.

Summing (5) and (7) we get

$$\begin{aligned} L_x(2G - A) &= 2G^2 + A^2 + \tfrac{1}{2}(P_x + \rho - 2P_{yz}) \\ &= (2G - A)^2 + 4AG - 2G^2 + \tfrac{1}{2}(P_x + \rho - 2P_{yz}) \\ &= (2G - A)^2 - 2P_x + \tfrac{1}{2}(P_x + \rho - 2P_{yz}) \end{aligned}$$

where in the last step we have used (6). Defining $K \equiv 2G - A$, this and the field equation become

$$\begin{aligned} L_x^2\phi - KL_x\phi - \partial_\phi V &= 0 \\ L_x K &= K^2 + 3V. \end{aligned}$$

Since $g_1 > 0$ there are no solutions in which K does not blow up. This also means that ϕ blows up. This is consistent with our earlier result and is independant of the choice of V as long as $V \geq 0$.

7 Time dependant domain walls

If we allow ϕ to become a function of time the stress-energy tensor picks up an off-diagonal term $L_t\phi L_x\phi$, so in the gauge where the stress-energy tensor is diagonal we have $L_t\phi = 0$. In the $g_2 = 0$ gauge this implies that ϕ is independant of time, but we are no longer free to choose $g_1 = 1$. Alternatively if we explictly look for solutions in which ϕ is not a function of time, and use the $g_2 = 0$ gauge, we see that the Stress-Energy tensor is automatically diagonal so we can still choose $g_1 = 1$. We shall do this for comparison with Widrow's work,[4] though he chose to use the $g_1 = f_1$ gauge.

Allowing the $\overline{h}$-function to become time dependant doesn't change the stress-energy tensor or the field equation. Since we choose $g_1 = 1$ we have $B = 0$. If we assume that α is separable we know $\partial_t G = 0$ and so $A = -G$ as before. We therefore have the two equations

$$L_x^2\phi - 3GL_x\phi - \partial_\phi V = 0$$
$$L_xG = G^2 + \frac{1}{3}[(L_x\phi)^2 + V].$$

The right hand side of the second equation is positive so G will become singular. This is independant of choice of V. Widrow[4] claims to have found good behaviour for the coupled equations with a particular choice of V, but he seems to have assumed that ϕ would tend to its uncoupled value, and has used a gauge in which the singularity would occur at infinity a finite proper distance away. As we can see ϕ does not converge to this value and there is a curvature singularity at a finite distance from the wall. The absence of a curvature singularity in the thin wall case depends critically on the absence of stress-energy, which is not true in any limiting approximation. In any case we have seen that a scalar field does not tend to its vacuum value.

8 Conclusions

We have seen that sensible plane symmetric solutions are in short supply. The study of domain walls shows that they must be time dependant with a time dependant ϕ.

References

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[3] Alexander Vilenkin. Gravitational field of vacuum domain walls. *Phys. Lett.* 1983 133B 177.

[4] Lawrence M. Widrow. General-relativistic domain walls. *Phys. Rev. D* 1989 39:2, 3571.