Notes on kink metrics in GTG

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Abstract

1 Introduction

Kink metrics are those in which the light cones tip over by an angle of $n\pi$, in the spherically symmetric case on the way from the origin to ∞. The spherically symmetric kink metric is not globally transformable to the Schwarzchild or Newtonian gauge since the transformation would have singularities. Locally it may be transformed, but the kink is a global property.

Hariott and Williams[1] have shown that the most general one kink spherically symmetric metric can be written

$$\begin{aligned}
g_{tt} &= e^{\lambda} - (e^{\lambda} + e^{\sigma})\sin^2\alpha \\
g_{tr} = g_{rt} &= (e^{\lambda} + e^{\sigma})\sin\alpha\cos\alpha \\
g_{rr} &= -e^{\sigma} + (e^{\lambda} + e^{\sigma})\sin^2\alpha \\
g_{\theta\theta} &= -e^{\gamma} r^2 \\
g_{\phi\phi} &= -e^{\gamma} r^2 \sin^2\theta
\end{aligned}$$

where α, λ, σ and γ are functions of r and t. For a one kink metric α is usually taken to satisfy

$$\alpha(r=0) = \pi \qquad \alpha(r=\infty) = 0. \tag{1}$$

Thee is the possibility of having $\alpha < 0$, which is equivalent to the time reverse of the metric with positive α. In GTG we work with the position gauge function $\overline{h}$ rather than the metric. There are four distinct $\overline{h}$-functions corresponding to the above metric. The two function which could become asymptotically Minkowskian is

$$\begin{aligned}
\overline{h}(e^t) &= u\cos\alpha\, e^t - v\sin\alpha\, e^r \\
\overline{h}(e^r) &= u\sin\alpha\, e^t + v\cos\alpha\, e^r \\
\overline{h}(e^\theta) &= we^\theta \\
\overline{h}(e^\phi) &= we^\phi
\end{aligned}$$

where u, v and w are functions of t and r and $u, v, w > 0$. This can be shown to to give the above metric by forming

$$g_{\mu\nu} = \underline{h}^{-1}(e_\mu)\cdot\underline{h}^{-1}(e_\nu)$$

and substituting

$$u = e^{-\lambda/2} \quad v = e^{-\sigma/2} \quad w = e^{-\gamma/2}.$$

In a recent paper Harriott and Williams[3] have studied the metric

$$ds^2 = e^{-2\Lambda}(e^{2\eta}\cos 2\alpha\, dt^2 + 2\sin 2\alpha\, dt\, dr - e^{-2\eta}\cos 2\alpha\, dr^2) - r^2 d\Omega^2$$

which is equivelent to the inverse of the above $\overline{h}$-function,

$$\begin{aligned}
\overline{h}(e^t) &= e^{\Lambda+\eta}(\cos\alpha\, e^t - \sin\alpha\, e^r) \\
\overline{h}(e^r) &= e^{\Lambda-\eta}(\sin\alpha\, e^t + \cos\alpha\, e^r) \\
\overline{h}(e^\theta) &= e^\theta \\
\overline{h}(e^\phi) &= e^\phi
\end{aligned}$$

This form of the $\overline{h}$-function is rather more intuitive than the metric. We clearly see that α is the angle of tilt of the light cones and that e^η measures the aspect ratio of the cones.

A problem with these metrics is that they have incomplete geodesics. As the light cones tip over there will inevitably be horizons at $\alpha = \pi/4$ and $\alpha = 3\pi/4$. In GR this problem is disposed of by extension, manipulating the manifold to make sure that the potentially terminating geodesic actually has somewhere to go. Unless we are going to allow non-simple manifolds in GTG there seems no way to solve the problem in GTG. There seems no a priori reason why we shouldn't have non-simple manifolds in GTG, except perhaps the simpler solution that such kink metrics are not valid solutions to Einstein's equations.

There exist solutions for the sperically symmetric kink which satisfy the weak, but not the strong, energy condition. Whether the corresponding pressure and density functions could correspond to anything physical is probably rather doubtful.

2 Solutions to Einstein's equation

Forgetting the above problems we now consider a cylindrical kink metric:

$$\begin{aligned}
\overline{h}(e^t) &= e^{\Lambda+\eta}(\cos\alpha\, e^t - \sin\alpha\, e^r) \\
\overline{h}(e^r) &= e^{\Lambda-\eta}(\sin\alpha\, e^t + \cos\alpha\, e^r) \\
\overline{h}(e^\phi) &= e^\phi \\
\overline{h}(e^z) &= e^z
\end{aligned}$$

To solve Einstein's equations we follow the method given in [4]. From the $\overline{h}$-function we form the covariant rotation gauge function

$$\omega(a) = -H(a) + \tfrac{1}{2} a\cdot(\partial_b \wedge H(b))$$

where

$$H(a) = -\overline{h}(\dot{\nabla})\wedge\dot{\overline{h}}\,\overline{h}^{-1}(a).$$

This assumes that the spin of the matter fields is zero. Using the kink $\overline{h}$-function this becomes

$$\begin{aligned}
\omega(e_t) &= e^{\Lambda-\eta}(\cos\alpha\, \partial_r[\Lambda+\eta] + \sin\alpha\, \partial_r\alpha)\, e_r e_t \\
\omega(e_r) &= e^{\Lambda-\eta}(\sin\alpha\, \partial_r[\Lambda+\eta] - \cos\alpha\, \partial_r\alpha)\, e_r e_t \\
\omega(e_\phi) &= e^{\Lambda-\eta}\sin\alpha\,/r\, e_\phi e_t + (1 + e^{\Lambda-\eta}\cos\alpha\,)/r\, e_\phi e_r \\
\omega(e_z) &= 0
\end{aligned}$$

We next find the covariant Riemann tenor

$$\mathcal{R}(a\wedge b) = L_a\omega(b) - L_b\omega(a) + \omega(a)\times\omega(b) - \omega\big(a\cdot\mathcal{D}b - b\cdot\mathcal{D}a\big)$$

where we have used the derivatives

$$L_a \equiv a\cdot\overline{h}(\nabla) \quad \text{and} \quad a\cdot\mathcal{D} \equiv L_a + \omega(a)\times$$

and then use the Ricci tensor and the Ricci scalar

$$\mathcal{R}(b) = \partial_a\cdot\mathcal{R}(a\wedge b) \quad \text{and} \quad \mathcal{R} = \partial_a\cdot\mathcal{R}(a)$$

to form the Einstein tensor

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}.$$

Einstein's equation is then

$$\mathcal{G}(a) = 8\pi\mathcal{T}(a)$$

where $\mathcal{T}$ is the covariant matter stess-energy tensor.

In the case where $\Lambda = 0$ it turns out that there is no way of getting a positive energy density for all α. In the more general case the Einstein tensor is not diagonal, and to see if we can rescue a positive energy density we diagonalize it by solving the eigenvalue equation

$$\mathcal{G}(a) - \lambda a = 0$$

or equivalently

$$\mathcal{G}'(i) = 0$$

where

$$\mathcal{G}'(a) = \mathcal{G}(a) - \lambda a.$$

We identify the energy density as the eigenvalue corresponding to the timelike eingenvector. It turns out to be

$$\rho = e^{2\Lambda}(M - L|\cos 2\alpha|)/r$$

where

$$M = -1/2\,\partial_r(\cos 2\alpha\, e^{-2\eta}) \quad \text{and} \quad L = e^{-2\eta}\partial_r\Lambda.$$

This can probably be made positive everywhere though it would require a very careful choice of functions.

References

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[4] Anthony Lasenby, Chris Doran and Stephen Gull. Gravity, Gauge Theories and Geometric Algebra. To appear in *Phil. Trans. R. Soc. Lon. A.*