Electroweak theory

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1 Notation and conventions

Bold letters are used for Pauli vectors except σ_k. Position gauge covariant variables are in caligraphic letters. This includes the derivatives, so that, acting on spinors,

$$D_a \equiv a\cdot\nabla + \tfrac{1}{2}\Omega(a) \qquad \mathcal{D}_a \equiv a\cdot\mathcal{D} \equiv a\cdot\overline{h}(\nabla) + \tfrac{1}{2}\omega(a)$$

Note that this notation differs from the GTG paper. The exception is the rotation gauge function whose covariant form is written $\omega(a) = \Omega\underline{h}(a)$.

We shall often be dealing with multivector valued linear functions of vectors. The traditional tensor notation appears to be well suited to dealing with these object, so we shall adapt this well established notation by defining

$$M_a \equiv M(a) \qquad M^a \equiv M(\partial_a).$$

For example, with a scalar valued linear function of a vector

$$A(a) = a\cdot V$$

we can form the square

$$A(a)A(\partial_a) = V\cdot V = A_a A^a.$$

This gives us a co-ordinate free notation reminiscent of Penrose's abstract index notation. Where we have a scalar valued function of a vector we can use the same letter for the linear function and it's representation in terms of a dot product with a vector

$$K_a = K(a) = a\cdot K.$$

Similarly we can use $\nabla_a = a\cdot\nabla$. We also adopt the convention that when the index notation it is assumed that the index is not a function of position. This notation can be extended to functions of bivectors, etc. For example

$$F_{a\wedge b} \equiv F(a\wedge b) \qquad F^{a\wedge b} \equiv F(\partial_a\wedge\partial_b) \qquad F_B \equiv F(B) \qquad F^B \equiv F(\partial_B).$$

2 Electroweak Theory

Since phase transformations of spinors have a useful representation in geometric algebra in terms of right multiplication of spinors it is interesting to see if we can construct a similar representation for electroweak theory. Here we build on the translation of the electroweak Lagrangian already developed[1], in particular we look for a translation of the Higgs mechanism and the gauged theory.

There is an obvious candidate Lagrangian for a spinor field with $SU(2)\times U(1)$ symmetry which is

$$\mathcal{L} = \langle \mathcal{D}\psi i\gamma_0\tilde{\psi}\rangle$$

where the $U(1)$ symmetry is generated by $e^{i\alpha}$ and the $SU(2)$ symmetry by right multiplication by $e^{i\sigma_k}$. Left handed fields form a doublet in electroweak theory which we translate as the single spinor

$$\psi \equiv \psi_e\tfrac{1}{2}(1-\sigma_3) - \psi_\nu i\sigma_2\tfrac{1}{2}(1+\sigma_3).$$

The right handed fields form singlets that are defined not to transform under $SU(2)$

$$\chi \equiv \psi_e\tfrac{1}{2}(1+\sigma_3).$$

These spinors can be used to write the kinetic part of the Lagrangian as

$$\mathcal{L} = -\langle \mathcal{D}\psi i\gamma_0\tilde{\psi} + \mathcal{D}\chi\gamma_0 i\tilde{\chi}\rangle.$$

We now introduce the weak isospin quantum number T and the weak hypercharge number Y. If a spinor ψ has weak isospin T and weak hypercharge Y then it transforms as

$$\psi \to \psi e^{i\alpha(T\sigma_3+Y)}.$$

We define the quantum numbers for ψ to be $Y = -\frac{1}{2}$ and $T = \frac{1}{2}$. The $U(1)_Q$ symmetry is generated by right multiplication of ψ_e and ψ_ν by $e^{\alpha i\sigma_3}$. Since the electron has charge one and the neutrino is chargeless the corresponding transformation on ψ is

$$\psi \to \psi e^{-i\alpha\frac{1}{2}(1-\sigma_3)}.$$

Acting on ψ the generator of the electric charge is therefore

$$Q = \tfrac{1}{2}\sigma_3 - \tfrac{1}{2}.$$

Generalising this to

$$Q = T\sigma_3 + Y$$

we can consistently give χ zero weak isospin and hypercharge 1. We now make the symmetries local by introducing the gauge fields $\boldsymbol{A}_a$, a Pauli vector valued function of a vector, and B, a vector field. The covariant derivative of a spinor ψ of weak hypercharge Y and weak isospin T becomes

$$D_a\psi = D_a^{\mathrm{old}}\psi - igT\psi\boldsymbol{A}_a - ig'YB_a\psi$$

where g and g' are the two coupling constants. The transformation laws for the gauge fields are

$$B \to B + \frac{1}{g'}\nabla\alpha$$

and

$$T\boldsymbol{A}_a \to e^{-i\boldsymbol{n}T}T\boldsymbol{A}_a e^{i\boldsymbol{n}T} - \frac{i}{g}e^{-i\boldsymbol{n}T}\nabla_a e^{i\boldsymbol{n}T}$$

Using the quantum number assignments we have the new covariant derivatives

$$\mathcal{D}\psi = \overline{h}^a(D_a\psi - \frac{ig}{2}\psi\boldsymbol{A}_a + \frac{ig'}{2}B_a\psi)$$

and

$$\mathcal{D}\chi = \overline{h}^a(D_a - g'B_a)\chi.$$

The commutator of the covariant derivatives gives rise to the non-Abelian field strength

$$\boldsymbol{F}_{a\wedge b} = \nabla_a \boldsymbol{A}_b - \nabla_b \boldsymbol{A}_a + ig\boldsymbol{A}_a \wedge \boldsymbol{A}_b$$

and the Abelian gauge field strength

$$I_{a\wedge b} = (a\wedge b)\cdot(\nabla\wedge B).$$

The Higgs mechanism gives mass to spinor fields through a three field interaction term of the form

$$-m_e(\psi\phi\tilde{\chi} + \chi\tilde{\phi}\tilde{\psi})$$

where

$$\phi = (a + bi\sigma_2)\tfrac{1}{2}(1-\sigma_3)$$

and a and b are 'complex' scalars, scalars plus pseudoscalars. This will be gauge invariant if the Higgs field ϕ has weak hypercharge $-\frac{1}{2}$ and under weak isospin transformations transforms as

$$\phi \to e^{-\frac{1}{2}i\boldsymbol{n}}\phi$$

i.e. that is is has weak isospin $-\frac{1}{2}$ and the transformation acts on the left. The full electroweak Lagrangian is now

$$\mathcal{L} = \langle -\mathcal{D}\psi i\gamma_0\tilde{\psi} + \mathcal{D}\chi i\gamma_0\tilde{\chi} - \tfrac{1}{4}\boldsymbol{\mathcal{F}}_B\cdot\boldsymbol{\mathcal{F}}^B - \tfrac{1}{4}\mathcal{I}_B\mathcal{I}^B + \nu^2(\mathcal{D}^a\phi)^\dagger\mathcal{D}_a\phi - 2m_e\psi\phi\tilde{\chi}\rangle$$

which has all the symmetries we require. We now proceed to invoke the Higgs mechanism by using the gauge freedom to set

$$\phi = \tfrac{1}{2}(1-\sigma_3).$$

This may be by the mechanism of spontaneous symmetry breaking or otherwise. In any case we shall for the moment assume that there are no excitations about this value, i.e. we shall ignore the physical Higgs field. The leptonic mass term now becomes

$$\mathcal{L}_{\text{lept},\phi} = -m_e\langle\psi(1-\sigma_3)\tilde{\chi}\rangle.$$

Now

$$D_a\phi = \frac{ig}{2}\boldsymbol{A}_a\tfrac{1}{2}(1-\sigma_3) + \frac{ig'}{2}B_a\tfrac{1}{2}(1-\sigma_3)$$

so writing $\boldsymbol{A}_a$ as

$$\boldsymbol{A}_a = A_{1a}\sigma_1 + A_{2a}\sigma_2 + A_{3a}\sigma_3$$

this becomes

$$D_a\phi = \frac{i}{2}\left[(g'B_a - gA_{3a}) + g(A_{1a} - iA_{2a})i\sigma_2\right]\tfrac{1}{2}(1-\sigma_3).$$

Defining the new fields

$$\begin{aligned} W_a &= \frac{1}{\sqrt{2}}(A_{1a} - iA_{2a}) && (1)\\ Z &= \cos\theta_W A_3 - \sin\theta_W B && (2) \end{aligned}$$

where

$$\tan\theta_W = g'/g$$

the Higgs part of the Lagrangian becomes

$$\mathcal{L}_{\text{Higgs}} = \langle\nu^2(\mathcal{D}^a\phi)^\dagger\mathcal{D}_a\phi\rangle = \frac{g^2\nu^2}{4}\left(\frac{\mathcal{Z}^2}{2\cos^2\theta_W} + \mathcal{W}^{a\dagger}\mathcal{W}_a\right).$$

There are therefore two physical massive fields, a charged vector field W_a of mass m_W and a neutral vector field Z of mass m_Z where

$$m_W^2 = \tfrac{1}{4}g^2\nu^2 \quad \text{and} \quad m_Z^2 = \frac{m_W^2}{\cos^2\theta_W},$$

and a massless vector field

$$A = \sin\theta_W A_3 + \cos\theta_W B.$$

We now define field strengths for the new fields so that

$$F_B^W = \frac{1}{\sqrt{2}}(\sigma_1 \cdot \boldsymbol{F}_B - i\sigma_2 \cdot \boldsymbol{F}_B) \tag{3}$$

$$\sigma_3 \cdot \boldsymbol{F}_{a\wedge b} = \sin\theta_W F_{a\wedge b}^A + \cos\theta_W F_{a\wedge b}^Z - ig(W_a W_b^\dagger - W_b W_a^\dagger) \tag{4}$$

The gauge part of the Lagrangian can now be written

$$\mathcal{L}_{\text{gauge}} = -\tfrac{1}{2}\mathcal{F}^{WB\dagger}\mathcal{F}_B^W + \tfrac{1}{2}\mathcal{F}^A \cdot \mathcal{F}^A + \tfrac{1}{2}\mathcal{F}^Z \cdot \mathcal{F}^Z \tag{5}$$

$$+i\mathcal{W}^a\mathcal{W}^{b\dagger}(e\mathcal{F}_{a\wedge b}^A + g\cos\theta_W \mathcal{F}_{a\wedge b}^Z) + \tfrac{1}{2}g^2[\mathcal{W}^a\mathcal{W}_a\mathcal{W}^{b\dagger}\mathcal{W}_b^\dagger - (\mathcal{W}^a\mathcal{W}_a^\dagger)^2] \tag{6}$$

where

$$e = g\sin\theta_W.$$

To see the interactions between the fields we rewrite the leptonic part of the Lagrangian in terms of the new variables

$$\mathcal{L}_{\text{lept}} = \mathcal{L}_{\text{kin.}} + \frac{g}{2\sqrt{2}}(\tilde{\mathcal{W}}\mathcal{J} + \mathcal{W}\tilde{\mathcal{J}}) - e\mathcal{A} \cdot \mathcal{J}^A + \frac{g}{2\cos\theta_W}\mathcal{Z} \cdot \mathcal{J}^Z$$

where

$$\mathcal{J} = \psi_e(1-\sigma_3)\gamma_0\tilde{\psi}_\nu \qquad \mathcal{J}^A = \psi_e\gamma_0\tilde{\psi}_e,$$

the neutral current is given by

$$\mathcal{J}^Z = \psi_\nu\tfrac{1}{2}(1-\sigma_3)\gamma_0\tilde{\psi}_\nu - \psi_e\tfrac{1}{2}(1-\sigma_3 - 4\sin^2\theta_W)\gamma_0\tilde{\psi}_e$$

and

$$\mathcal{W} = \overline{h}^a W_a$$

is a vector+trivector. We now have the final Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{lept.}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{lept},\phi}.$$

The generalisation to quarks is straightforward. We are left with a translation scheme which seems to have little to recommend it over the conventional matrix representation, though the implementation via internal symmetries may be significant. In any case one can still see clearly the inelegance of much of electroweak theory: we have to assume particle charges and masses, assume a doublet Higgs field, its transformation law and coupling constants, and assume the transformation laws for the left and right handed fields.

References

[1] Chris Doran. The Electroweak Lagrangian. 1996, unpublished.