Scale Invariant Gauge Theory Gravity

Antony Lewis, etc.

December 2, 1998

Abstract

We study the introduction of a local Weyl scaling symmetry in the context of gauge theory gravity. Gauging arguments lead to the introduction of a vector gauge field. We introduce a scalar field and construct a conformally invariant Lagrangian. The field equations are derived and gauge fixing discussed. This recovers standard gauge theory gravity with the addition of a massive vector field that only couples to other scalar fields. We compare our approach to corresponding theories in GR and mention possible extensions.

1 Introduction

The idea of a locally conformally invariant theory of gravity goes back a long way. Shortly after Einstein's theory was developed Weyl proposed an extension of Riemannian geometry in which vectors not only change direction on parallel displacement around a closed loop but also change their length. Weyl's non-Riemannian geometry was the first example of a gauge theory, the gauge group being local conformal transformations. To enforce this new gauge symmetry one has to introduce a vector gauge field which Weyl identified with the vector potential of electromagnetism. Weyl's theory was a bold, though ultimately unsuccessful attempt to provide a geometric picture of electromagnetism in the same way that general relativity (GR) provided a geometric picture of gravity. The theory was rejected because it seemed to conflict with the absolute length scales appearing in quantum phenomena.

Dirac[1] took up Weyl's theme as a way of implementing his Large Numbers Hypothesis. He thought the gravitational constant could change with time in order to keep the dimensionless number formed from fundamental constants approximately equal to the age of the universe expressed in atomic units. To this end his gravitational Lagrangian coupled a scalar field to the Ricci scalar, forming a correctly scale invariant Lagrangian.

We now know that local conformal invariance is not the symmetry that generates electromagnetism; however that does not mean that a conformal gauge symmetry should be rejected out of hand. The subject has since been studied by various authors, for instance see [2]-[9]. Here we examine the symmetry in gauge theory gravity (GTG) as recently formulated by Lasenby *et al.*[10] GTG is a modernized version of ECKS or U_4 spin-torsion theory where gravity corresponds to a combination of invariance under local Lorentz transformations and diffeomorphisms. GTG reproduces the results of GR for all the standard tests, but also incorporates torsion in a natural manner.

In the gauge theory approach to gravity all observables must be gauge invariant, in particular they must be conformal gauge invariant. The result of a measurement is a dimensionless number, we do not measure dimensionful quantities. Since observables are merely ratios of dimensionful quantities there is a global invariance under rescaling of all lengths (and masses, etc, appropriately). One could equally well regard this as invariance under change of units. Whether one prefers the passive or active viewpoint is largely a matter of taste, though it seems more natural to consider physics resulting from invariance under active transformations. The idea is to make this global scaling symmetry a local symmetry by the introduction of a gauge field (the "Weylon"). In the original formulation it was thought that local standards of length would lead to 'non-integrability of length', a highly non-desirable feature of Weyl's geometric approach. In the gauge theory approach it is clear that this is not the case. Observable ratios of lengths are integrable and the theory becomes acceptable.

We employ the Spacetime Algebra, the Geometric Algebra of Minkowski spacetime over which gauge fields are defined. We follow the Spacetime Algebra notation and conventions of [10] with one exception explained below. For details of Geometric Algebra the reader is referred to [11, 10]. Natural units are used throughout ($\hbar = c = G = 1$).

2 Gauge Theory Gravity

Here we briefly re-cap on the gauging arguments of GTG. A fuller account can be found in [10].

In field theories field equations are derived from an action principle. For example the massless Dirac equation arises from minimization of

$$S = \int |d^4x| \langle \nabla \psi i\gamma_3 \tilde{\psi} \rangle.$$

As it stands this is not invariant under local displacements (diffeomorphisms) where

$$x \to f(x).$$

The first gauge principle of gravity is that the action should be invariant under local displacements of this kind. Clearly this is automatic for scalar and spinor fields if they are undifferentiated since $\phi'(x) = \phi(x')$. However the gradient with respect to $x' = f(x)$ is

$$\nabla_{x'} = \overline{f}^{-1}(\nabla_x, x)$$

where we have used the inverse of the adjoint of $\underline{f}(a,x) = a\cdot\nabla_x f(x)$. To ensure that derivatives transform in the same way as the fields we define the displacement gauge function $\overline{h}$ such that

$$\overline{h}'(a,x) = \overline{h}(\overline{f}^{-1}(a,x), f(x)).$$

The displacement covariant derivative now transforms as we require

$$\overline{h}(\nabla_x, x) \to \overline{h}(\nabla_{x'}, x').$$

From now on we shall drop the explicit x-dependence. The local displacement invariant Dirac action is now

$$S = \int |d^4x| \det \underline{h}^{-1} \langle \overline{h}(\nabla) \psi i\gamma_3 \tilde{\psi} \rangle.$$

To see whether the $\overline{h}$-field has physical effects consider the displacement covariant derivative in the a direction by

$$L_a = a\cdot\overline{h}(\nabla).$$

The commutator of these derivatives gives

$$[L_a, L_b] = \underline{h}^{-1}(L_a\underline{h}(b) - L_b\underline{h}(a))\cdot\overline{h}(\nabla).$$

Now

$$[L_a\underline{h}(b) - L_b\underline{h}(a)]\cdot c = (b\wedge a)\cdot(\overline{h}(\nabla)\wedge\overline{h}(c)).$$

so the condition that the commutator is zero is

$$\overline{h}(\nabla)\wedge\overline{h}(c) = 0 \qquad \Longrightarrow \qquad \nabla\wedge\overline{h}^{-1}(c) = 0.$$

Since this in general will not be the case the $\overline{h}$ field has physical effects as well as ensuring reparameterization invariance of the action.

The displacement invariant action is also invariant under rotations (active Lorentz transformations) of the fields where

$$\psi \to R\psi \qquad \overline{h}(a) \to R\overline{h}(a)\tilde{R}$$

and R is a rotor. We make the rotation invariance local by defining the bivector rotation gauge function $\Omega(a)$ that transforms as

$$\Omega(a) \to R\Omega(a)\tilde{R} - 2a\cdot\nabla R\tilde{R}.$$

This is used to define the derivative

$$D_a\psi \equiv (a\cdot\nabla + \tfrac{1}{2}\Omega(a))\psi \qquad D_a\overline{h}(b) \equiv a\cdot\nabla\overline{h}(b) + \Omega(a)\times\overline{h}(b)$$

where the form of the derivative depends on whether it is acting on a spinor or a spacetime multivector. The notation for the covariant derivatives is the only place in which our notation here differs from that used in [10]. The commutator of the rotation gauge covariant derivatives defines the Riemann tensor $R(a\wedge b)$:

$$[D_a, D_b]M \equiv R(a\wedge b)\times M = (a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b))\times M$$

We also can impose displacement covariance by defining

$$\mathcal{D} \equiv \overline{h}(\partial_a)D_a \qquad \mathcal{D}_a = a\cdot\mathcal{D} = a\cdot\overline{h}(\partial_b)D_b.$$

where we are adopting the useful convention that covariant (displacement gauge covariant) objects are written in calligraphic type. We also define the covariant Riemann tensor

$$\mathcal{R}(a\wedge b) = R(\underline{h}(a\wedge b))$$

and its contractions the Ricci tensor and the Ricci scalar

$$\mathcal{R}(b) = \partial_a\cdot\mathcal{R}(a\wedge b) \qquad \mathcal{R} = \partial_a\cdot\mathcal{R}(a).$$

3 Gauging scale invariance

Above we have summarized the symmetries of GTG as formulated by Lasenby *et al.*[10] We now extend the gauged symmetries to include local scale transformations. Every field is assigned a conformal weight depending on how it transforms under scaling:

$$\overline{h}(a) \to e^{\lambda}\overline{h}(a) \qquad M \to e^{n\lambda}M$$

where a quantity M has weight n. Spinors have weight $3/2$ as can be seen from the form of the Dirac Lagrangian and the Lagrangian is automatically locally scale invariant as can easily be checked. However when we have derivatives of a scalar field we need to define a new covariant derivative

$$\underline{D}_a \equiv D_a + na\cdot K$$

where under the scaling K transforms as

$$K \to K - \nabla\lambda.$$

One could introduce a coupling constant but it can be absorbed into the definition of K. The new covariant derivative satisfies Leibnitz law and scales as

$$\underline{D}_a M \to e^{n\lambda}\underline{D}_a M.$$

The transformation law for the K field is the same as that for the photon, which was what lead Weyl to an identification. Here we regard K as a new field associated with new physics since we now know that it is not conformal symmetry that generates electromagnetism.

In GR the Ricci scalar does not transform homogeneously and can be combined with derivatives of a scalar field to produce a combination which is overall invariant:

$$\phi^2\mathcal{R} + 6\partial_\mu\phi\partial^\mu\phi.$$

Such invariant terms without the K field are not an option in the gauge theory approach. Here the Riemann tensor transforms homogeneously at the level of the action, so we must introduce a new gauge field if we want an invariant kinetic term for the scalar field.

The commutator of the scale covariant derivative gives

$$[\underline{D}_a, \underline{D}_b]\phi = (a\cdot\nabla b\cdot K - b\cdot\nabla a\cdot K)\phi = (b\wedge a)\cdot(\nabla\wedge K)\phi.$$

where ϕ is a scalar field of weight one. We therefore define the field strength

$$H \equiv \nabla\wedge K.$$

The full derivative that's covariant under displacements is formed in the usual way

$$\underline{\mathcal{D}} = \overline{h}(\partial_a)\underline{D}_a.$$

We now proceed to construct a gauge invariant action integral of the form

$$S = \int |d^4x| \det \underline{h}^{-1}\mathcal{L}$$

where $\mathcal{L}$ must have weight four. As for electromagnetism we can form a kinetic term in the K-field is proportional to

$$\mathcal{H}\cdot\mathcal{H} = \overline{h}(H)\cdot\overline{h}(H).$$

Since $\det\underline{h}$ and $\mathcal{H}\cdot\mathcal{H}$ have weight four this term is conformal gauge invariant. However the Ricci scalar has conformal weight two, so this cannot be in the Lagrangian on its own. We therefore follow Dirac[1] by introducing a scalar field ϕ of conformal weight one and coupling its square to the Ricci scalar. The total Lagrangian is now

$$\mathcal{L} = \frac{\nu^2}{2}\underline{\mathcal{D}}\phi\cdot\underline{\mathcal{D}}\phi - \frac{\alpha}{2}\phi^2\mathcal{R} - \Lambda\phi^4 + \frac{\beta^2}{2}\mathcal{H}\cdot\mathcal{H} + \mathcal{L}_m$$

where $\mathcal{L}_m$ is the Lagrangian of any other fields. There are other terms quadratic in the Riemann that could be introduced but these make things much more complicated. To determine the classical field equations we now vary the action with respect to the four independent fields $\overline{h}(a)$, $\Omega(a)$, K and ϕ. This procedure is consistent with the gauge theory formulation and leads to results which differ somewhat from those obtained within extensions of GR.

4 The Field Equations

The field equations are derived by using the multivector function form of the Euler-Lagrange equation:

$$\partial_{M(a)}(\mathcal{L}\det \underline{h}^{-1}) = \partial_b\cdot\nabla(\partial_{M(a),b}\mathcal{L}\det \underline{h}^{-1})$$

where as usual $M(a)_{,b}$ denotes $b\cdot\nabla M(a)$.

Since $\overline{h}(a)$ is undifferentiated in the action the $\overline{h}(a)$ equation is

$$\begin{aligned} & \partial_{\overline{h}(a)}\mathcal{L} = 0 \\ \implies & \alpha\phi^2\mathcal{G}(a) = \mathcal{T}(a) \end{aligned} \tag{1}$$

where $\mathcal{G}(a)$ is the Einstein tensor defined by

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}$$

and $\mathcal{T}(a)$ is the stress-energy tensor defined by

$$\mathcal{T}\underline{h}^{-1}(a) = \det \underline{h}\partial_{\overline{h}(a)}([\mathcal{L} + \tfrac{1}{2}\alpha\phi^2\mathcal{R}]\det \underline{h}^{-1}).$$

This is the obvious extension of the $\overline{h}$-equation obtained in [10]. Similarly variation with respect to $\Omega(a)$ gives the modified 'wedge' equation

$$\alpha(\underline{\mathcal{D}} + \mathcal{U})\wedge(\phi^2\overline{h}(a)) = S(a)$$

where $S(a)$ is the 'spin' of the matter

$$S(a) = \partial_{\Omega(a)}\mathcal{L}_m$$

and

$$\mathcal{U} = \det \underline{h}D_b[\overline{h}(\partial_b)\det \underline{h}^{-1}] = \underline{h}^{-1}(a)\cdot[\underline{\mathcal{D}}\wedge\overline{h}(\partial_a)].$$

In the case where the spin is of Dirac type the contraction of the spin tensor vanishes[10] so that

$$\begin{aligned} & \underline{h}^{-1}(a)\cdot[(\underline{\mathcal{D}} + \mathcal{U})\wedge(\phi^2\overline{h}(\partial_a))] = 0 & (2)\\ \implies & 2\phi^2\mathcal{U} + 3\underline{\mathcal{D}}\phi^2 = 0 & (3)\\ \implies & \mathcal{U} = -3\underline{\mathcal{D}}\phi\phi^{-1}. & (4) \end{aligned}$$

We shall assume this is the case, and can therefore write the field equation as

$$\alpha\phi^2(\underline{\mathcal{D}} - \underline{\mathcal{D}}\phi\phi^{-1})\wedge\overline{h}(a) = \mathcal{S}(\overline{h}(a)). \tag{5}$$

Variation with respect to K gives

$$\begin{aligned} \nu^2\underline{h}(\phi\underline{\mathcal{D}}\phi) + \partial_K\mathcal{L}_m &= \tfrac{1}{2}\det \underline{h}\beta^2\partial_a\cdot\nabla(\partial_{K,a}[(\nabla\wedge K)\cdot\underline{h}(\mathcal{H})]\det \underline{h}^{-1}) & (6)\\ &= -\beta^2\det \underline{h}\nabla\cdot[\underline{h}(\mathcal{H})\det \underline{h}^{-1}]. & (7) \end{aligned}$$

Rearranging this becomes

$$\nu^2\phi\underline{\mathcal{D}}\phi = -\beta^2\underline{h}^{-1}(\nabla\cdot G)\det \underline{h} - \underline{h}^{-1}(\partial_K\mathcal{L}_m) \tag{8}$$

where $G = \underline{h}(\mathcal{H})\det \underline{h}^{-1}$. Finally the ϕ equation gives

$$\begin{aligned} -4\phi^3\Lambda - \alpha\phi\mathcal{R} + \partial_\phi\mathcal{L}_m &= -\nu^2\mathcal{K}\cdot\underline{\mathcal{D}}\phi + \nu^2\det \underline{h}\partial_a\cdot\nabla[\partial_{\phi,a}(\det \underline{h}^{-1}\phi_{,b}\overline{h}(\partial_b))\cdot\underline{\mathcal{D}}\phi] & (9)\\ &= -\nu^2\mathcal{K}\cdot\underline{\mathcal{D}}\phi + \nu^2(\mathcal{U} - \Omega(a)\times\overline{h}(\partial_a) + 3\mathcal{K} + \overline{h}(\nabla))\cdot\underline{\mathcal{D}}\phi & (10)\\ &= \nu^2(\underline{\mathcal{D}} + \mathcal{U})\cdot\underline{\mathcal{D}}\phi & (11) \end{aligned}$$

Using (4) this becomes

$$4\Lambda + \alpha\phi^{-2}\mathcal{R} - \phi^{-3}\partial_\phi\mathcal{L}_m = -\nu^2\underline{\mathcal{D}}\cdot(\phi^{-3}\underline{\mathcal{D}}\phi).$$

and we can use use (1) to substitute for $\mathcal{R}$ giving

$$-4\Lambda + \phi^{-4}\mathcal{T} + \phi^{-3}\partial_\phi\mathcal{L}_m = \nu^2\underline{\mathcal{D}}\cdot(\phi^{-3}\underline{\mathcal{D}}\phi). \tag{12}$$

where $\mathcal{T} = \partial_a\cdot\mathcal{T}(a)$.

In [10] there was considerable analysis of the quantity $\mathcal{U}$. In the absence of the conformal symmetry the field equations implied that it vanished if the spin tensor had zero contraction. This was just what was required for the minimal coupling procedure to be consistent at both the level of the action and the field equations. Here the Dirac Lagrangian is automatically locally scale invariant. It gives rise to the field equation

$$\underline{\mathcal{D}}\psi i\sigma_3 = m\psi\gamma_0 - \tfrac{1}{2}\mathcal{U}\psi i\sigma_3$$

where the K in the derivative and in $\mathcal{U}$ can be cancelled from both sides. If the contraction of the spin tensor vanishes $\mathcal{U} = -3\underline{\mathcal{D}}\phi\phi^{-1}$. Instead of the minimally coupled equation

$$\underline{\mathcal{D}}\psi i\sigma_3 = m\psi\gamma_0$$

we find that the local conformal invariance is maintained by the addition of a ϕ term:

$$\mathcal{D}\psi i\sigma_3 = m\psi\gamma_0 + \frac{3}{2}\mathcal{D}\phi\phi^{-1}\psi i\sigma_3. \tag{13}$$

Interestingly we cannot find any action that would give rise to the first equation. In the case where ϕ is fixed to one this all reduces to the same as in [10].

5 The Einstein gauge

We can recover the equations of GTG by defining the conformally invariant variables

$$\overline{h}'(a) = \overline{h}(a)/\phi \quad \text{and} \quad S'(a) = S(a)/\phi^4. \tag{14}$$

With these definitions (5) becomes

$$\mathcal{D}'\wedge\overline{h}'(a) = \kappa S'(a)$$

where $\kappa = 1/\alpha$. If $\alpha = 1/8\pi G$ this is the 'wedge' equation of GTG. Similarly we can recover Einstein's equation by using the new invariant variables

$$\mathcal{G}(a)' = \phi^{-2}\mathcal{G}(a) \quad \text{and} \quad \mathcal{T}(a)' = \phi^{-4}\mathcal{T}(a)$$

in equation (1). Observables must be gauge invariant and we can guarantee this by using the dashed variables. We can set the dashed variables equal to the undashed variables by fixing the conformal gauge so that $\phi = 1$, called the Einstein gauge since it reduces the $\overline{h}$ equation to Einstein's equation. This is possible in the presence of stress-energy since ϕ must be non-zero and the Lagrangian is symmetric under $\phi \to -\phi$. However a more general gauge would be needed to discuss the possibility of $\phi = 0$ and at the quantum level we anticipate other gauges being more useful. With $\phi = 1$ we see that the quartic ϕ term in the action would give rise to a cosmological constant if $\Lambda \neq 0$ (see [12] for a discussion).

Using (5) and fixing $\phi = 1$ in the K-equation (8) we have

$$\nu^2\mathcal{K} + \underline{h}^{-1}(\partial_K\mathcal{L}_m) = -\beta^2\underline{h}^{-1}(\nabla\cdot G)\det\underline{h} \tag{15}$$

$$= -\beta^2[\mathcal{D}\cdot\mathcal{H} - \mathcal{S}(\partial_a)\cdot(a\wedge\mathcal{H})]. \tag{16}$$

When the spin is a trivector (as for electroweak spinors) the spin $\mathcal{S}(a) = \mathcal{S}\cdot a$ where $\mathcal{S} = \psi i\gamma\tilde{\psi}$ and γ is a basis vector. In this case we can write

$$\underline{h}^{-1}(\partial_K\mathcal{L}_m) + \nu^2\mathcal{K} = -\beta^2(\mathcal{D}\cdot\mathcal{H} - \kappa\mathcal{S}\cdot\mathcal{H}). \tag{17}$$

Taking the divergence and making use of the relation[10]

$$\mathcal{D}\wedge\overline{h}(A_r) = \overline{h}(\nabla\wedge A_r) + \kappa\langle\mathcal{S}\overline{h}(A_r)\rangle_{r+1}$$

we get

$$\nu^2\mathcal{D}\cdot\mathcal{K} + \mathcal{D}\cdot\overline{h}^{-1}(\partial_K\mathcal{L}_m) = -\beta^2(i\overline{h}(\nabla\wedge\overline{h}^{-1}(i\mathcal{D}\cdot\mathcal{H})) - \kappa\mathcal{D}\cdot(\mathcal{S}\cdot\mathcal{H})).$$

Now

$$i\mathcal{D}\cdot\mathcal{H} = -\overline{h}(\nabla\wedge\overline{h}^{-1}(i\mathcal{H})) + \kappa i\mathcal{S}\cdot\mathcal{H}$$

so that

$$i\overline{h}(\nabla\wedge\overline{h}^{-1}(i\mathcal{D}\cdot\mathcal{H})) = \kappa\mathcal{D}\cdot(\mathcal{S}\cdot\mathcal{H})$$

and finally

$$\nu^2\mathcal{D}\cdot\mathcal{K} = -\mathcal{D}\cdot\underline{h}^{-1}(\partial_K\mathcal{L}_m). \tag{18}$$

Since K can only enter the matter Lagrangian via derivatives of scalar fields we see that K is divergenceless in the absence of other scalar fields. To see what sort of field K is we can write equation (17) as

$$\mathcal{D}^2\mathcal{K} - \mathcal{D}(\mathcal{D}\cdot\mathcal{K}) - \bar{\mathcal{R}}(\mathcal{K}) - \kappa(\mathcal{D} - \kappa\mathcal{S})\cdot(\mathcal{K}\cdot\mathcal{S}) + m^2\mathcal{K} = 0$$

where $m = \nu/\beta$. When $\mathcal{L}_m$ is independent of K and in the absence of torsion this is the equation of a massive vector field of mass m:

$$\mathcal{D}^2\mathcal{K} - \bar{\mathcal{R}}(\mathcal{K}) + m^2\mathcal{K} = 0$$

Since K only couples to scalar fields directly it is essentially non-interacting and therefore makes a potential dark matter candidate.[3]

The mass here differs from that found in Weyl's geometry where the only gravitational field in the action is the metric. This difference in masses is typical of the result obtained when varying one rather than two fields in the action.[13] In GR this is equivalent to the difference between the Palatini principle, where the connection is varied as an independent field, versus variation with respect to the metric where the connection is predetermined. In the gauge theory approach it is clear that $\Omega(a)$ should be varied independently of $\overline{h}(a)$. The difference in predicted masses is probably trivial since the theory doesn't predict the value of β (unless perhaps $\beta = 1$). However the difference would be much more radical with a more complicated (e.g. quadratic) Lagrangian.

Fixing $\phi = 1$ and using Einstein's equation the ϕ equation (12) becomes

$$\nu^2\mathcal{D}\cdot\mathcal{K} - \partial_\phi\mathcal{L}_m|_{\phi=1} - \mathcal{T}^{\mathcal{L}_m} = 0.$$

Substituting for $\mathcal{D}\cdot\mathcal{K}$ from (18) we have the equation

$$\partial_\phi\mathcal{L}_m|_{\phi=1} + \mathcal{T}^{\mathcal{L}_m} + \mathcal{D}\cdot\underline{h}^{-1}(\partial_K\mathcal{L}_m) = 0 \tag{19}$$

that must be satisfied by any locally conformally invariant Lagrangian. We note that the trace of the stress-energy tensor is not zero, as expected for a realistic theory.[12]

In summary, we have recovered GTG spin-torsion theory but with an additional massive vector field that doesn't couple to spinors. This is very similar to conformal extensions of GR. However the gauge theory $\Omega(a)$ field is weight zero while the connection in GR contains derivatives of the metric tensor and must be modified to contain Ks. This is the key difference between the gauge theory formulation and the introduction of local conformal invariance in GR via a new gauge field. It explains why we get different results from the field equations, though it's not clear that there any observable differences. The gauge theory approach would seem to be more consistent.

6 Discussion

Tsamis and Woodward[12] have shown that some scalar-metric theories of GR are identical to their pure metric analogs. The theories they analysed were generated by forming conformally invariant combinations of the Ricci scalar and derivatives of the scalar field, and the action is

$$\bar{S}[\phi, g, \psi_i] = S[\phi^2 g, \phi^{-n}\psi_i]$$

where ψ_i are fields with weight n (we have defined weights to be the negative of what they are usually defined to be). Similarly they show that the quantum theory in the Einstein gauge is identical to the pure metric theory. They conclude that local conformal invariance is a sham conformal symmetry without dynamical import.

In our approach much of their argument goes through. For instance equations (14) and (13) show that ϕ is still really just a scale factor for the other fields. However since we have to introduce a new gauge field there does seems to be some new physics. The bare theory we end up with is identical to the non-conformal theory with the addition of a minimally coupled massive vector field. The ϕ field is still without dynamical import, but its kinetic term can't be generated directly from the non-conformal Lagrangian by substituting $\overline{h}(a) = \phi^{-1}\overline{h}(a)$ as it could be in GR. We now have the K field which transforms inhomogeneously so all we can say is that

$$\bar{S}[\phi, \overline{h}(a), \Omega(a), \psi_i, K] = S[\phi^{-1}\overline{h}(a), \Omega(a), \phi^{-n}\psi_i, K + \phi^{-1}\nabla\phi],$$

which is easily proved by performing a scaling proportional to ϕ^{-1}. Similar remarks of course apply to theories in GR where a new gauge field is introduced.

In order to generate a more physically interesting theory it is possible to consider whether ϕ might have some internal structure. A tempting possibility is identifying ϕ with the modulus of the Higgs field. The standard model Lagrangian can be written

$$\mathcal{L} = \langle -\mathcal{D}\psi i\gamma_0\tilde{\psi} + \mathcal{D}\chi i\gamma_0\tilde{\chi} - \tfrac{1}{4}\boldsymbol{\mathcal{F}}_{ab}\cdot\boldsymbol{\mathcal{F}}^{ab} + \tfrac{1}{2}\mathcal{I}\cdot\mathcal{I} + \nu^2(\mathcal{D}^a\phi_H)^\dagger\mathcal{D}_a\phi_H - 2m_e\psi\phi_H\tilde{\chi}\rangle - V$$

which is conformally invariant if V is. (The details of our translation into geometric algebra are not important here). To fit in with conformal gravity the electroweak Lagrangian must be locally scale invariant. This means that the covariant derivatives must be replaced with the scale covariant invariant ones and that V must be of weight four. There are two possibilities. We could have

$$V = \eta(\phi_H^2 - \nu_H^2\phi^2)^2$$

to recover the standard model when we use the Einstein gauge. Alternatively we could identify the modulus of the Higgs field with ϕ in which case we must have $V = \Lambda\phi^4$. In this case there is no symmetry breaking; instead the symmetry remains unbroken and the observed particle spectrum comes from working in the Einstein gauge. Of course observables must be gauge invariant so this provides an interesting alternative to the usual Higgs mechanism. The main prediction is that the Higgs particle doesn't exist since there is now one extra degree of freedom — the conformal gauge freedom. The ϕ field can now be set exactly to one. This also means that K doesn't couple to any standard model fields except via gravity.

This Higgs free model has been developed before.[7, 6] If we don't adopt the Higgs free model we gain some terms in the standard model from the coupling of K to the Higgs field. This has the advantage of making the K field potentially detectable, and allows standard model physics to remain otherwise unchanged (though we haven't checked the quantum implications).

7 Conclusions

We have shown that it is possible to develop a locally conformally invariant gauge theory of gravity that is compatible with all current observations. Indeed in the absence of other scalar fields the theory has been shown to be identical to simply adding a massive vector field to the non-conformal theory. The only physical interest therefore seems to come from considering the new field as a potential dark matter candidate, everything else is the same.

With the addition of other scalar fields things may be more interesting. It is possible to identify the ϕ field with the modulus of the Higgs field, giving the bold prediction that the Higgs particle doesn't exist. There is a good possibility that this may be discounted shortly by the discovery of a Higgs, in which case physical interest enters through the coupling of the K field to the Higgs. This could potentially give a mechanism for generating dark-matter Ks in the early universe.

The conformal formalism may also have mathematical uses. The possibility of not using the Einstein gauge gives a different mathematical approach to essentially the same physical problem. One could consider fixing the conformal gauge by fixing K rather than ϕ, which could have applications in constructing a quantum theory.

References

[1] P.A.M. Dirac. Long range forces and broken symmetries. *Proc. Roy. Soc. (Lo ndon)*, A333:403, 1973.

[2] M. Israelit. Massive electrodynamics and magnetic monopoles. *GR and Grav.*, 29(11):1411, 1997.

[3] M. Israelit and N. Rosen. Weyl-Dirac geometry and dark matter. *Found. Phys.*, 22(4):555, 1992.

[4] N. Rosen. Weyl's geometry and physics. *Found. Phys.*, 12(3):213, 1982.

[5] D. Gregorash and G. Papini. Weyl-Dirac theory with torsion. *Nuovo. Cim.*, 55B:37, 1980.

[6] M. Pawłowski and R. Rączka. A Higgs-free model for fundamental interactions. *Found. Phys.*, 24:1305, 1994.

[7] H. Cheng. Possible existence of Weyl's vector meson. *Phys. Rev. Lett.*, 61(19):2182, 1988.

[8] W. Drechsler and H. Tann. Broken Weyl-invariance and the origin of mass. *gr-qc/9802044*, 1998.

[9] J. Solà. The cosmological constant and the fate of the cosmon in Weyl conformal gravity. *Phys. Lett. B*, 228(3):317, 1989.

[10] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Gravity, gauge theories and geometric algebra. *Phil. Trans. R. Soc. Lond. A*, 356:487–582, 1998.

[11] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus.* Reidel, Dordrecht, 1984.

[12] N. Tsamis and R. Woodward. No new physics in conformal scalar-metric theory. *Ann. Phys.*, 168:257, 1986.

[13] U. Lindström. The Palatini variational principle and a class of scalar-tensor theories. *Nuovo Cim.*, 35B:130, 1976.