Igloo notes

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Abstract

I describe the changes in the igloo code for polarization, and summarise what the computation is doing a bit more explicitly (and, hopefully, correcting typos).

1 Introduction

Generalization of the results in [1] to polarization is mostly very straightforward. We use the complex spin two polarization

$$Q \mp iU = \sum_{lm} a_{\pm 2,lm\pm 2}Y_{lm} = \sum_{lm}(E_{lm} \pm iB_{lm})_{2\pm}Y_{lm}.$$

The negative frequency harmonics coefficients can be related to the positive ones using

$$E_{l(-m)} + iB_{l(-m)} = (-1)^m (E_{lm} - iB_{lm})^*.$$

The pixelized values are

$$(Q \mp iU)_p \equiv \int d\Omega W_p (Q \mp iU) = {}_{\pm 2}W^p_{lm} a_{\pm 2,lm}$$

where the window function W_p (zero outside and $1/A_p$ inside the pixel for pixel averages, delta function for pixel centres) defines the pixelized harmonics

$${}_sW^p_{lm} = \int d\Omega W^p {}_sY_{lm}.$$

A pixelized spin s field ${}_sX$ has harmonic coefficients

$$a^{\rm pix}_{s,lm} \equiv \sum_p A_p {}_sX^p {}_sW^{p*}_{lm}.$$

We only consider pixelizations where all the pixels in each row have the same width, and the pixelization is symmetric about the equator (the former to do fast FFTs, the latter to halve the computation time - though not essential). Let q label the pixel in each row, with $N_p - 1 \geq q \geq 0$ and r now label the row. The first pixel in each row starts at $\phi = \phi_0$. For pixel averaging we have

$$\begin{aligned} {}_sW^{rq}_{lm} &= \frac{1}{A_{rq}} \int_{2\pi q/N_r+\phi_0}^{2\pi(q+1)/N_r+\phi_0} d\phi e^{im\phi} \int_{\theta_t}^{\theta_b} d\theta \sin\theta {}_s\lambda_{lm} & (1) \\ &= e^{2\pi i m q/N_r} {}_sW^r_{lm} & (2) \end{aligned}$$

where

$$ {}_sW^r_{lm} = e^{im\phi_0} e^{im\pi/N_r} \frac{\sin \pi m/N_r}{\pi m/N_r} \frac{1}{\Delta x_r} {}_sI^r_{lm} $$

and we have defined

$$ {}_sI^r_{lm} \equiv \int_{\theta_\mathrm{t}}^{\theta_\mathrm{b}} {}_s\lambda_{lm} \sin\theta d\theta_r = \int_{x_b}^{x_t} {}_s\lambda_{lm} dx $$

and $\Delta x_r = |\Delta \cos\theta|$. The pixelized data can be generated using

$$ {}_sX_{rq} = \sum_{m=-l_\mathrm{max}}^{l_\mathrm{max}} {}_sb^r_m e^{2\pi iqm/N_r} = \sum_{m=-l_\mathrm{max}}^{l_\mathrm{max}} {}_sb^r_m e^{2\pi iq \mod (m,N_r)/N_r} $$

where

$$ {}_sb^r_m \equiv \sum_{l=\max(|m|,|s|)}^{l_\mathrm{max}} a_{s,lm} \, {}_sW^r_{lm}. $$

For a realistic map l_max should be chosen large enough to include all of the power in the sky, and aliasing into lower modes is then an aspect of the pixelization error.

The expression for the pixelized data

$$ {}_sX_{rq} = {}_sb^r_0 + \sum_{m=1}^{l_\mathrm{max}} \left[{}_sb^r_m e^{2\pi iq \mod (m,N_r)/N_r} + {}_sb^r_{-m} e^{2\pi iq \mod (N_r-m,N_r)/N_r} \right] \tag{3} $$

$$ = \sum_{m=0}^{N_r-1} \left[\sum_{\mod (m',N_r)=m} {}_sb^r_{m'} + \sum_{\mod (N_r-m',N_r)=m, m'\neq 0} {}_sb^r_{-m'} \right] e^{2\pi iqm/N_r} \tag{4} $$

can be evalutaed by FFT. Writing ${}_sb^r_m{}^{(\mathrm{tot})}$ for the sum in the square brackets one can invert the transform to give for all m

$$ {}_sb^r_m{}^{(\mathrm{tot})} \equiv \frac{1}{N_r} \sum_{q=0}^{N_r-1} {}_sX_{rq} e^{-2\pi iqm/N_r} = \frac{1}{N_r} \sum_{q=0}^{N_r-1} {}_sX_{rq} e^{-2\pi iq \mod (m,N_r)/N_r}. $$

The pixelized harmonic coefficients are given by

$$ a^\mathrm{pix}_{s,lm} \equiv \sum_{rq} A_r \, {}_sX_{rq} \, {}_sW^{rq*}_{lm} = \sum_r N_r A_r \, {}_sb^r_m{}^{(\mathrm{tot})} \, {}_sW^{r*}_{lm} = 2\pi \sum_r \Delta x_r \, {}_sb^r_m{}^{(\mathrm{tot})} \, {}_sW^{r*}_{lm}, $$

where the area of each pixel in row r is $A_r = 2\pi\Delta x_r/N_r$.

Ref.[1] suggests the estimator

$$ a^\mathrm{est}_{s,lm} = \frac{a^\mathrm{pix}_{s,lm}}{N_{slm}}, $$

where

$$ N_{slm} \equiv \sum_r N_r A_r |{}_sW^r_{lm}|^2 = 2\pi \sum_r \Delta x_r |{}_sW^r_{lm}|^2. $$

For the polarization we stick with spin $+2$ harmonics, so

$$(E_{lm}+iB_{lm})^{\rm est} = \frac{2\pi}{N_{2lm}}\sum_r \Delta x_r\, {}_2b^{r\ ({\rm tot})}_m\, {}_2W^{r*}_{lm}$$

$$(E_{lm}-iB_{lm})^{\rm est} = (-1)^m\frac{2\pi}{N_{2lm}}\sum_r \Delta x_r\, {}_2b^{r*\ ({\rm tot})}_{-m}\, {}_2W^{r}_{l(-m)}$$

where we have used $N_{2lm} = N_{2l(-m)}$ which follows from ${}_sI_{lm}(\text{south}) = (-1)^{l+s}{}_sI_{l(-m)}(\text{north})$. This symmetry can also be used to halve the computation by using the symmetry of the north and south hemispheres [1].

2 Pixel integrals

The integral of spin ± 2 harmonics can be obtained recursively by integration of the recursion relations give in [2] for $l > |m|$. The $l = |m|$ integrals can be related to those of the spin zero harmonics that need to be computed for the temperature map. Defining

$$C_{slm} \equiv \sqrt{\frac{l^2(4l^2-1)}{(l^2-m^2)(l^2-s^2)}} \tag{5}$$

the relations needed for $s = 2$, for $l \geq |m| \geq 2$ are

$$\frac{(l+2)}{C_{s(l+1)m}}{}_sI_{(l+1)m} = \frac{sm}{l(l+1)}{}_sI_{lm} + \frac{(l-1)}{C_{slm}}{}_sI_{(l-1)m} - \left[(1-x^2)_s\lambda_{lm}\right] \qquad l > |m|$$

$${}_sI_{mm} = \frac{2}{m-s+2}\left[(1+x)_s\lambda_{mm}\right] + \sqrt{\frac{(m+s)(m-s+1)}{(m+s-1)(m-s+2)}}{}_{(s-2)}I_{mm}$$

$${}_sI_{m(-m)} = \frac{2}{m-s+2}\left[(x-1)_s\lambda_{m(-m)}\right] + \sqrt{\frac{(m+s)(m-s+1)}{(m+s-1)(m-s+2)}}{}_{(s-2)}I_{m(-m)}$$

and otherwise ${}_2I_{l0} = {}_0I_{l2}$ and

$${}_2I_{21} = -\frac{1}{6}\sqrt{\frac{5}{16\pi}}\left([2+(3-2x)x]\sqrt{1-x^2} + 3\sin^{-1}x\right)$$

$${}_2I_{2(-1)} = -\frac{1}{6}\sqrt{\frac{5}{16\pi}}\left([2+x][2x-1]\sqrt{1-x^2} + 3\sin^{-1}x\right).$$

3 Cut sky

It is very straightforward to stop the map analysis at some $\theta_{\rm cut}$, giving zero contribution from the rest of the sky. However the question remains about how to estimate the pseudo harmonic coefficients. In particular N_{slm} depends on the entire pixelization. I have assumed that since one is not doing any computation over the cut sky one wants to tile it optimally using infinitely small pixels. The contribution to N_{slm} from the cut can then be computed analytically using the relations given in [2] for the inner product of spin weight harmonics. This may not be the best thing to do.

References

[1] R. Crittenden and N. Turok. Exactly azimuthal pixelizations of the sky. 1998.

[2] A. Lewis, A. Challinor, and N. Turok. Analysis of CMB polarization on an incomplete sky. Accepted by PRD, 2001.