Scalar fields in ekpyrosis

AML

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Abstract

Things I did to convince myself of some results in Neil's paper. Real space solutions. Other potentials, solutions, non-linear effects.

1 Introduction

Many predictions of ekpyrotic scenarios can be worked out using an effective 4D theory involving a scalar field with a potential[1, 2, 3]. A convenient model is

$$V(\phi) = V_0(1 - e^{-c\phi})F(\phi)$$

where $c \approx 10{\rm M_{pl}}^{-1}$and $V_0 \approx 10^{-120}{\rm M_{pl}}^4$. Perturbations are generated in the collapsing phase where $F(\phi) \approx 1$ and t approaches zero from below (i.e. the time before the Big Bang). To a first approximation the gravitation effects can be ignored during perturbation generation, so the universe is approximately Minkowskian of constant size and $\frac{1}{2}\dot{\phi}^2 + V \approx 0$. Assuming the exponential dominates, so $V = -V_0 e^{-c\phi}$ one has the background solution

$$\phi = \frac{2}{c}\ln(-t\sqrt{\tfrac{1}{2}V_0 c^2})$$

and $V_{,\phi} = 2/ct^2$. The perturbation equation is

$$\ddot{\delta\phi} = \nabla^2\delta\phi + \frac{2}{ct^2}(1 - e^{-c\delta\phi})$$

which is a function only of $f \equiv c\delta\phi$. In linear theory there is the solution

$$f_k \propto \frac{1}{\sqrt{k}} e^{\pm ikt}\left(1 \pm \frac{i}{kt}\right)$$

just like in inflation, giving a scale invariant spectrum as $t \to 0$.

Other potentials

The ekpyrotic mechanism is an interesting, and fairly generic, alternative to inflation-driven perturbation generation. The exponential potential in the above approximations leads to an exactly scale invariant spectrum. Other potentials will give something slightly different. For example, for $V(\phi) = -(a/\phi)^n$, one obtains $\phi = ((2+n)^2 a^n t^2/2)^{1/(n+2)}$, giving the equation

$$\ddot{\delta\phi} + k^2\delta\phi = \frac{n(n+1)}{(2+n)^2}\frac{2\delta\phi}{t^2}.$$

The solution is

$$(-kt)^{1/2}\left[C_1 J_{-\alpha_n/2}(-kt) + C_2 J_{\alpha_n/2}(-kt)\right]$$

where $\alpha_n \equiv \sqrt{1+8n(n+1)/(2+n)^2}$. As $t \to -\infty$ one has

$$\frac{A}{2}\sqrt{\frac{\pi}{2k}}\frac{i-1}{\sin(\pi\alpha_n/2)}(-kt)^{1/2}\left[e^{-i\alpha_n}J_{\alpha_n/2}(-kt) - e^{i\alpha_n}J_{-\alpha_n/2}(-kt)\right] \to \frac{A}{\sqrt{2k}}e^{-ikt}.$$

The solution that grows as $t \to 0$ is $(-kt)^{1/2}J_{-\alpha_n/2}(-kt)$ which diverges as $(-kt)^{(1-\alpha_n)/2}$. Therefore at small times one has

$$\delta\phi \approx \frac{A}{2}\sqrt{\frac{\pi}{2k}}\frac{i-1}{\sin(\pi\alpha_n/2)}\frac{2^{\alpha_n/2}e^{-i\alpha_n}(-kt)^{(1-\alpha_n)/2}}{\Gamma(1-\alpha_n/2)}.$$

This implies a spectral index $n_s = 4 - \alpha_n$, which is scale invariant for large n.

It would have been easier to use the Hankel functions $J_v(-kt) \pm iY_v(-kt)$ which have the large argument limit

$$J_v(x) \pm iY_v(x) \to \sqrt{\frac{2}{\pi x}}e^{\mp i\pi(1+2v)/4}e^{\mp ix}$$

and zero limit

$$J_v(x) \pm iY_v(x) \to \frac{\mp i x^{-v}2^v}{\sin(\pi v)\Gamma(1-v)}.$$

Any evolution equation solved by Hankel functions will therefore give rise to a power spectrum characterised exactly by a spectral index. In general however k dependence would be more complicated and the spectral index would run.

Real space solutions

Spherically-symmetric scaling solutions are obtained by assuming f is a function only of $x \equiv -r/t$, when the equation becomes

$$(x^2-1)f'' + (2x - 2/x)f' = 2(1-e^{-f}).$$

There is an analytic spherically-symmetric scaling solution to the linearized theory

$$f \propto 2 + \frac{(x^2-1)}{x}\ln\left|\frac{x-1}{x+1}\right|$$

which goes as $1/x^2$ as $x \to \infty$ and is well behaved everywhere except for a discontinuity in the second derivative at $x = 1$. The solution corresponds to shrinking perturbations that become narrower and narrower but with constant value in the centre. I have generated corresponding non-linear solutions numerically (the strategy being to use series solutions at 0 and ∞ and propagate towards $x = 1$, using a binary search to find the initial values at which they meet up), which behave qualitatively the same way, though the region in which the derivatives become equal at $x = 1$ becomes smaller and smaller, making the solutions look rather odd and the numerics rather difficult. The solutions are unstable to small perturbations in the initial amplitude (I tried adding a small (1%) Gaussian overdensity which diverges from the scaling solution very rapidly), and are probably of very limited physical significance.

If $\nabla^2\delta\phi$ can be neglected there is the solution

$$f = \ln\left[\frac{2}{t}\cos([t-d]e^{-2A})\right]^2 + A$$

where A and d are integration constants. This solution diverges to $+\infty$ as $t \to 0$, i.e. perturbation is left an infinite distance behind as the background runs to minus infinity. There are singularities at $t = d + (\pi/2 + n\pi)e^{2A}$, as the perturbation runs to minus infinity before the background.

Otherwise one can propagate the initial value problem numerically using a particular realization of a perturbation distribution. Large negative perturbations grow more negative rather fast, and it is essentially to cut-off the exponential (i.e. with $F(\phi)$) to get finite answers, which can nonetheless be very negative. Large regions of negative perturbation correspond to bubbles that grow at close to the speed of light.

Gaussianity

Because the perturbations are small they are expected to be Gaussian to good accuracy when they leave the horizon. The subsequent evolution is governed by the background evolution and should not affect the Gaussianity.

Exact solutions with gravity

For amusement we include some exact solutions for a single scalar field and gravity (may be useful for constructing realistic solutions from exact pieces, inflation, etc).

The standard Ekyprotic exponential:

$$V = -\frac{p(1-3p)}{t^2} = -p(1-3p)e^{2(b-\phi)/\sqrt{2p}} \qquad \phi = \sqrt{2p}\ln(-t) + b \qquad a = (-t)^p$$

Others:

$$\begin{aligned}
V &= b^2 p\left[(3p-1)\cosh^2((\phi - c)/\sqrt{2p}) + 1\right] \\
&= b^2 p\frac{[3p\cosh^2(bt) - 1]}{\sinh^2(bt)} \\
\phi &= -2\sqrt{2p}\tanh^{-1} e^{bt} + c = \sqrt{2p}\ln\tanh(-bt/2) + c \\
a &= [-\sinh(bt)]^p \\
H &= pb\coth(bt)
\end{aligned}$$

For $p = 1/3$ this is a solution for $V = \text{const}$! Similar:

$$\begin{aligned}
V &= pb^2 \sinh^2((c-\phi)/2\sqrt{2p})((3p-1)\sinh^2((c-\phi)/2\sqrt{2p})-1) \\
&= pb^2 \frac{(3pe^{bt}-1)}{\sinh^2(bt/2)} \\
\phi &= \sqrt{2p}\ln\tanh(-bt/4)+c \\
a &= [1-e^{bt}]^p \\
H &= \frac{pb}{1-e^{-bt}}
\end{aligned}$$

There are equivalent trigonometric solutions. Some of these are really useless, but anyway..

$$\begin{aligned}
V &= \frac{b^2B^2}{3} + \frac{b^2}{16}\phi^2(3\phi^2 - 8(B+1)) \\
\phi &= -2\sqrt{\frac{d}{b}}e^{bt} \\
H &= \frac{bB}{3} - de^{2bt}
\end{aligned}$$

i.e. roll very slowly off top of hill and set of for $-\infty$.

It's a bit tricky working out the conformal time for all but the exponential potential.

Tensors

The equation is

$$h'' + 2\mathcal{H}h' = -k^2 h$$

and for the exponential potential $\mathcal{H} = p/\tau(1-p)$. The solution is

$$\frac{\sqrt{-k\tau}}{a}\left[AH_v^+(-k\tau) + BH_v^-(-k\tau)\right]$$

where $v = (1-3p)/2(1-p) \approx 1/2$. So to match to the vacuum we want

$$h = \frac{\sqrt{\pi}}{2a}\sqrt{-\tau}H_v(-k\tau)$$

which at small times gives

$$h \sim k^{-v}$$

So they are not amplified and $n_t = 3 - 2v = 2/(1-p)$ so the spectrum is blue. It is obviously nearly $n_t = 2$ because $\mathcal{H}$ has to be small to get the scalar spectrum right, and the equation is then just that of a free field so we are missing a power of k in h.

Note that h is like the curvature perturbation. The variable $s \equiv a^2\sigma \equiv -a^2h'/k$ is like the density perturbation and is finite at the singularity (so σ blows up). It obeys the equation

$$s'' - 2\mathcal{H}s' = -k^2 s$$

and $h = s'/ka^2$.

Bounce

There are some problems getting through the bounce. The background variables expand as

$$w \equiv \frac{p}{\rho} = 1 + w_1\tau + w_2\tau^2 + w_3\tau^3 + \dots$$

$$a^2 \propto \tau - \frac{3w_1}{8}\tau^2 + \dots$$

$$\mathcal{H} = \frac{1}{2\tau} - \frac{3}{16}w_1 + (\frac{9}{128}w_1^2 - \frac{1}{8}w_2)\tau + \dots$$

$$c_s^2 \equiv \frac{p'}{\rho'} = 1 + \frac{2}{3}w_1\tau + \left(\frac{w_2}{3} + \frac{1}{24}w_1^2\right)\tau^2 + \dots$$

$$t = \tau^{3/2}(1 - 9w_1/80\tau + \dots)$$

The series solutions are

$$E(\tau) = \tau^2 + \frac{5}{6}w_1\tau^3 + \dots$$

$$D(\tau) = 1 + \frac{3}{4}w_1\tau + d_3\tau^3 + \dots - \tfrac{1}{2}k^2E(\tau)\ln\tau$$

which depend on w which is different on either side. ($d_3 = w_3/4 - 3w_1/16[k^2 + w_2 + 5/8w_1^2]$). Note D'' diverges. Matching is problematic [4], and there may be problems with particle production [5]. Does writing it this way help disambiguate the relative normalization of the solutions?

The final result for the exponential potential matched onto a $V = 0$ potential at $V = V_{\min}$ is

$$k^{3/2}\chi \approx \frac{|V_{\min}|^{1/2}}{6}\left(\frac{k}{k_0}\right)^{-p} k_0^{-p}\zeta^2 e^{4\phi_{\min}/\sqrt{6}}$$

where k_0 is the pivot scale ($1\text{Mpc} \approx 4 \times 10^{56}L_{pl}$), which introduces a significant numerical factor unless $p \ll 1/50$. Note that for different potentials one may well have $n_s > 1$, in which case, for $n_s \approx 1.1$ one can gain an $\mathcal{O}(10^{-3})$ factor this way. Since the effect of gravity is to redden one may get such a result from $1/\phi^n$ where n is not so large (without gravity need $n \approx 40$).

The tensor perturbations are log divergent at the bounce. As for the scalars this is because the effective 4D action is log divergent there.

References

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