Non-Gaussianity notes

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Derive the optimal estimator from likelihood of isotropized sky. Fisher errors. Effect of ISW.

I. INTRODUCTION

A statistically-isotropic and parity-invariant CMB bispectrum $B_{l_1 l_2 l_3}$ is defined by

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle \equiv B_{l_1 l_2 l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (1)$$

$$= b_{l_1 l_2 l_3} \int \mathrm{d}\Omega Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3}, \quad (2)$$

where $b_{l_1 l_2 l_3}$ is the reduced bispectrum. If T_{lm} is a Gaussian field, then

$$T'_{lm} = T_{lm} + \frac{1}{6} B_{l l_1 l_2} \sum_{m_1 m_2} (-1)^{m_1} \begin{pmatrix} l & l_1 & l_2 \\ m & -m_1 & m_2 \end{pmatrix} T_{l_1 m_1} T^*_{l_2 m_2}$$

will have bispectrum $B_{l_1 l_2 l_3}$ to lowest order in the assumed small non-Gaussianity.

An anisotropy estimator from an observed sky is

$$6 X_{lm} \equiv \sum_{l_1 m_1, l_2 m_2} B_{l l_1 l_2} (-1)^{m_1} \begin{pmatrix} l & l_1 & l_2 \\ m & -m_1 & m_2 \end{pmatrix} \bar{\Theta}_{l_1 m_1} \bar{\Theta}^*_{l_2 m_2}$$

$$= \int \mathrm{d}\Omega Y^*_{lm} \times \sum_{l_1 l_2} b_{l l_1 l_2} \left[\sum_{m_1} \bar{\Theta}_{l_1 m_1} Y_{l_1 m_1} \right] \left[\sum_{m_2} \bar{\Theta}_{l_2 m_2} Y_{l_2 m_2} \right]. \quad (3)$$

Subtracting the zero-mean part from the observed sky gives

$$\Theta^G_{lm} \equiv \Theta_{lm} - f_{\rm NL}(X_{lm} - \langle X_{lm} \rangle),$$

where $f_{\rm NL}$ is an unknown non-Gaussianity amplitude. Now if the anisotropy is due to a primordial bispectrum, we find that the field Θ^G_{lm} has no leading-order bispectrum. So

$$-2 \log P(\mathbf{\Theta}^G) \sim \mathbf{\Theta}^{G\dagger} \boldsymbol{C}^{-1} \mathbf{\Theta}^G + \text{const.}$$

where $\boldsymbol{C} = \boldsymbol{C}^{TT} + \boldsymbol{N}$, and we assume the power spectrum is known. [ignore problems with higher order terms] Then

$$P(\mathbf{\Theta}) = P(\mathbf{\Theta}^G) \left| \frac{\partial \mathbf{\Theta}^G}{\partial \mathbf{\Theta}} \right|.$$

The maximum likelihood satisfies $\partial_{f_{\rm NL}} \log P(\mathbf{\Theta}) = 0$ so

$$[\mathbf{\Theta} - f_{\rm NL}(\mathbf{X} - \langle \mathbf{X} \rangle)]^\dagger \boldsymbol{C}^{-1} (\mathbf{X} - \langle \mathbf{X} \rangle) = \mathrm{Tr}\left[(\boldsymbol{I} - f_{\rm NL} \mathrm{d}\mathbf{X}/\mathrm{d}\mathbf{\Theta})^{-1} \partial \mathbf{X}/\partial \mathbf{\Theta} \right]. \quad (4)$$

The leading Newton-Raphson solution is then

$$\mathcal{E} = \frac{1}{F_\mathcal{E}} \left\{ \bar{\mathbf{\Theta}}^\dagger (\mathbf{X} - \langle \mathbf{X} \rangle) - \mathrm{Tr}\left[\partial \mathbf{X}/\partial \mathbf{\Theta} \right] \right\} \quad (5)$$

$$= \frac{1}{F_\mathcal{E}} \bar{\mathbf{\Theta}}^\dagger (\mathbf{X} - 3\langle \mathbf{X} \rangle), \quad (6)$$

and

$$F_\mathcal{E} = (\mathbf{X} - \langle \mathbf{X} \rangle)^\dagger \boldsymbol{C}^{-1} (\mathbf{X} - \langle \mathbf{X} \rangle) + \mathrm{Tr}\left[\partial \mathbf{X}/\partial \mathbf{\Theta} \partial \mathbf{X}/\partial \mathbf{\Theta} \right].$$

Approximating

$$F_\mathcal{E} \sim \langle F_\mathcal{E} \rangle = 3 \mathrm{Tr}\left[\boldsymbol{C}^{-1} \mathrm{cov}(\mathbf{X}) \right], \quad (7)$$

this is the optimal estimator for weakly non-Gaussian fields [1–3]

$$\mathcal{E} = \frac{1}{6 F_\mathcal{E}} \sum_{l_i m_i} B_{l_1 l_2 l_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \times \left[\bar{\Theta}_{l_1 m_1} \bar{\Theta}_{l_2 m_2} \bar{\Theta}_{l_3 m_3} - 3 C^{-1}_{l_1 m_1 l_2 m_2} \bar{\Theta}_{l_3 m_3} \right]. \quad (8)$$

This is just the leading $f_{\rm NL}$ term in the expansion of the likelihood divided by the expectation of the second term; the form can also be read off easily from the leading Edgeworth expansion (three derivatives of a Gaussian w.r.t. $\bar{\Theta}_{lm}$). On the full sky with isotropic noise

$$\frac{1}{\sigma^2_{f_{\rm NL}}} = F_\mathcal{E} = \frac{1}{6} \sum_{l_1 l_2 l_3} \frac{B^2_{l_1 l_2 l_3}}{C_{l_1} C_{l_2} C_{l_3}}.$$

Local non-Gaussianity gives small scale anisotropies correlated with the large scale fluctuations. Observationally the largest-scale modes are a combination of sources at last scattering and more local ISW contributions. If we could estimate the fluctuations at last scattering, we would expect these to be better correlated with the observed small scale anisotropy, and hence obtain a higher signal-to-noise estimate of the non-Gaussianity. For local non-Gaussianity

$$B_\chi(k_1, k_2, k_3) = \pm 2 \frac{3}{5} f_{NL} (P_\chi(k_1) P_\chi(k_2) + 2 \text{ perms.}) \quad (9)$$

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*URL: `http://cosmologist.info`

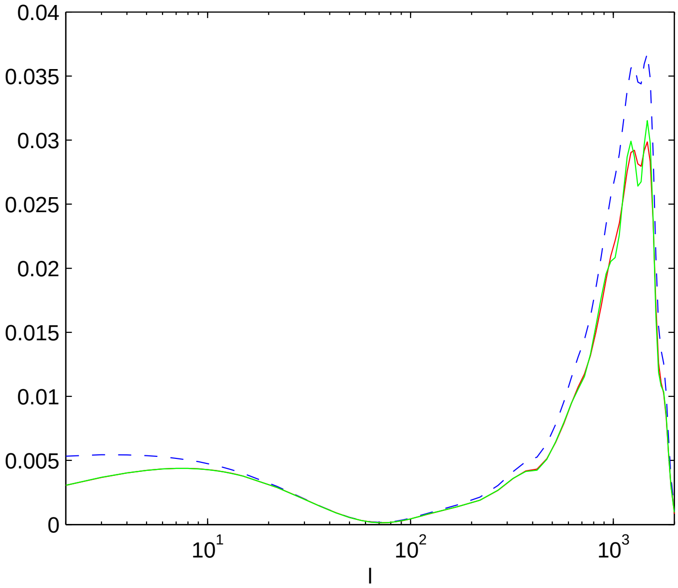

FIG. 1: Contributions to the TT Fisher inverse variance on $f_{\rm NL}$ for a local model, $l\sum_{l_2 l_3} B^2_{l l_2 l_3}/(6C_l C_{l_2} C_{l_3})$, with (solid) and without (dashed) ISW contributions, assuming a simple isotropic-noise full sky with Planck-like parameters. Green is using the lensed power spectra.

where the 3/5 is conventional (from relation the curvature perturbation to the Newtonian potential in the radiation dominated era) and $\pm$ is a sign convention. The reduced bispectrum is then

$$b_{l_1 l_2 l_3} = \pm\frac{3}{5} f_{NL}\int r^2 {\rm d}r \beta_{l_1}(r)\beta_{l_2}(r)\alpha_{l_3}(r) + 5 \text{ perms.} \tag{10}$$

To assess the scope for improving estimates by removing ISW, one can simply compare the Fisher errors for the observed CMB compared to the observed CMB without ISW contributions. Note that removing ISW also has the effect (if it could be done!) of removing the leading ISW-CMB lensing bispectrum. Removing ISW would seem to decrease the TT-only error bar by about 20%.

If we measure a field $\mathbf{d}$ with $C^{d-{\rm ISW}}$ correlation to the ISW part of the CMB, then $\hat{\Theta}_{{\rm ISW},lm} = \Theta_{lm} - C_l^{d-{\rm ISW}} d_{lm}/C_l^{dd}$ is an estimate of the ISW-cleaned temperature. The variance $\langle(\Theta_{\rm ISW}-\hat{\Theta}_{\rm ISW})^2\rangle = [\boldsymbol{C}^{-1}]^{-1}_{\Theta\Theta}$ where $\boldsymbol{C}$ is the full covariance of the maps. Combining multiple density maps with different redshift window functions, we can significantly reduce the ISW component, see figure. CMB lensing also happens to have a very similar kernel to the ISW (with correlations of over 90% between the lensing potential and the ISW part of the temperature), and potentially does well on its own. The right way to actually estimate non-Gaussianity from combined CMB and other maps is equivalent to doing CMB with correlated polarization [4].

The ISW-lensing signal comes from a small number of

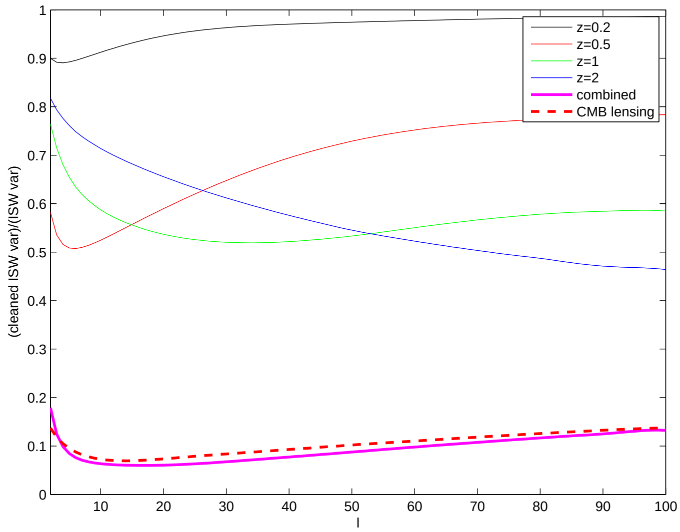

FIG. 2: Reduction in ISW variance when subtracting using combinations of number counts in various redshift window functions. Gaussian widths are $\Delta z = \{0.1, 0.2, 0.2, 0.5\}$. CMB lensing result is assuming perfect knowledge of lensing potential. ISW is defined here as the contribution from $z < 20$, i.e. not including the early effect. Combined uses four counts windows, but not including CMB lensing.

large scale models, say $l \lesssim 10$, so might expect the variance on the signal to go like $1/10^2$, so $\sim 10\%$ error on $f_{\rm NL}$ with $\langle f_{\rm NL}\rangle \sim \mathcal{O}(10)$, which is significantly smaller than the error from the Gaussian temperature variance, so not much to be gained by removing ISW-lensing correlation if the mean contribution can be calculated accurately.

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