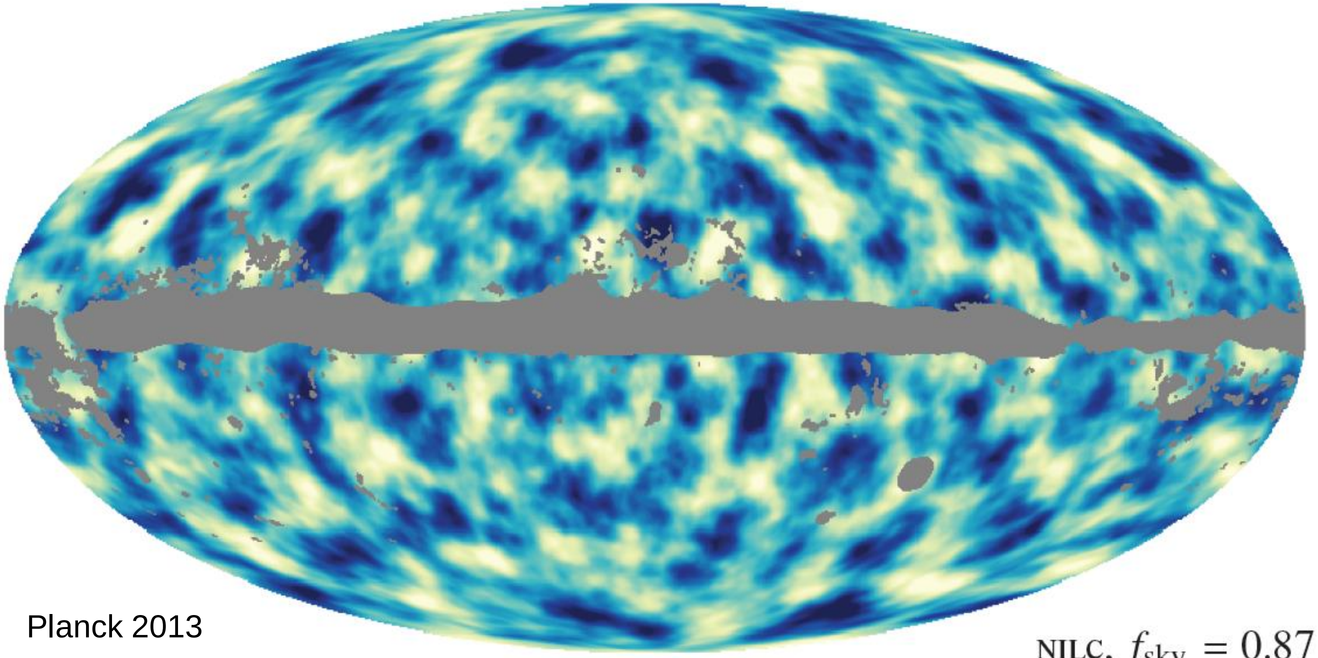
Lensing & ISW

Antony Lewis



Planck 2013

NILC, $f_{\rm sky} = 0.87$

University of Sussex

http://cosmologist.info/

Physics Reports review: **astro-ph/0601594**

Weak Gravitational Lensing of the CMB

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Abstract

Weak gravitational lensing has several important effects on the Cosmic Microwave Background (CMB): it changes the CMB power spectra, induces non-Gaussianities, and generates a B-mode polarization signal that is an important source of confusion for the signal from primordial gravitational waves. The lensing signal can also be used to help constrain cosmological parameters and lensing mass distributions. We review the origin and calculation of these effects. Topics include: lensing in General Relativity, the lensing potential, lensed temperature and polarization power spectra, implications for constraining inflation, non-Gaussian structure, reconstruction of the lensing potential, delensing, sky curvature corrections, simulations, cosmological parameter estimation, cluster mass reconstruction, and moving lenses/dipole lensing.

Key words: Cosmic Microwave Background; Gravitational Lensing
PACS: 98.80.Es, 98.70.Vc, 98.62.Sb, 8.80.Hw

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Weak lensing of the CMB

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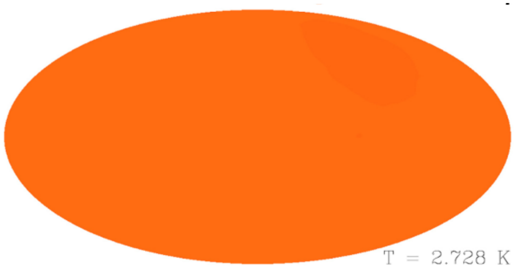
Abstract The cosmic microwave background (CMB) represents a unique source for the study of gravitational lensing. It is extended across the entire sky, partially polarized, located at the extreme distance of $z = 1100$, and is thought to have the simple, underlying statistics of a Gaussian random field. Here we review the weak lensing of the CMB, highlighting the aspects which differentiate it from the weak lensing of other sources, such as galaxies. We discuss the statistics of the lensing deflection field which remaps the CMB, and the corresponding effect on the power spectra. We then focus on methods for reconstructing the lensing deflections, describing efficient quadratic maximum-likelihood estimators and delensing. We end by reviewing recent detections and observational prospects.

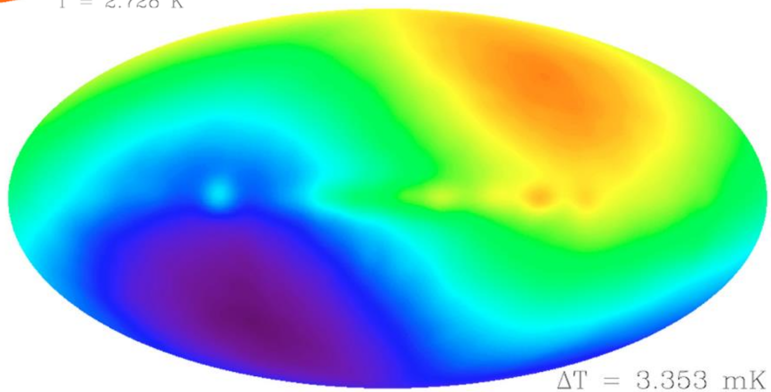
Outline

- Order of perturbations
- Light propagation, linear CMB and ISW
- Non-linear effects
- Lensing order of magnitudes
- Lensed power spectrum

- CMB polarization
- Reconstructing the potential
- Cosmological parameters
- Non-Gaussianity
- Cluster lensing

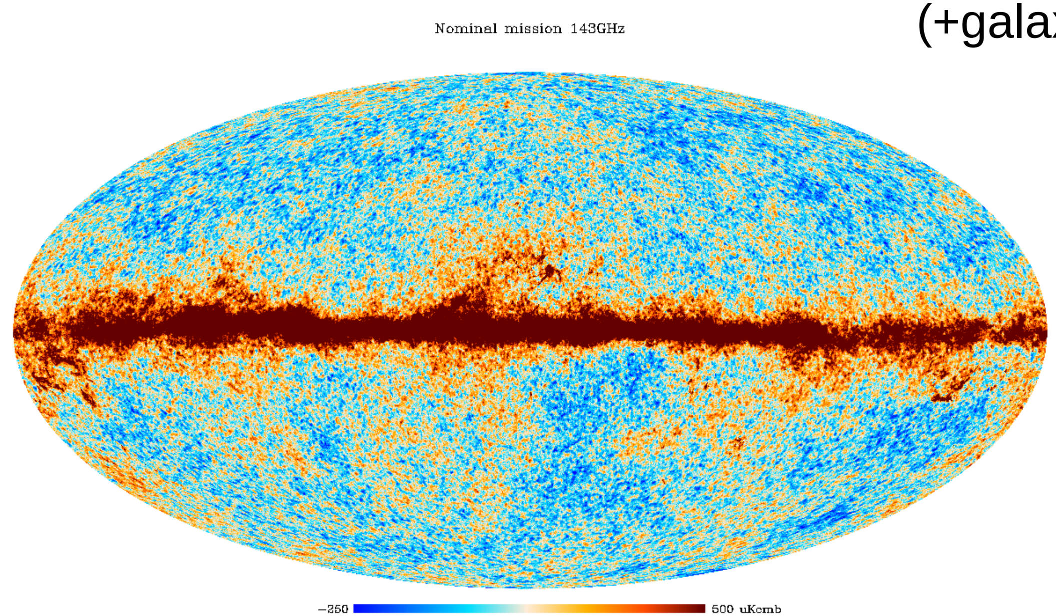


(almost) uniform 2.726K blackbody



Dipole (local motion)

O(10^{-5}) perturbations
(+galaxy)



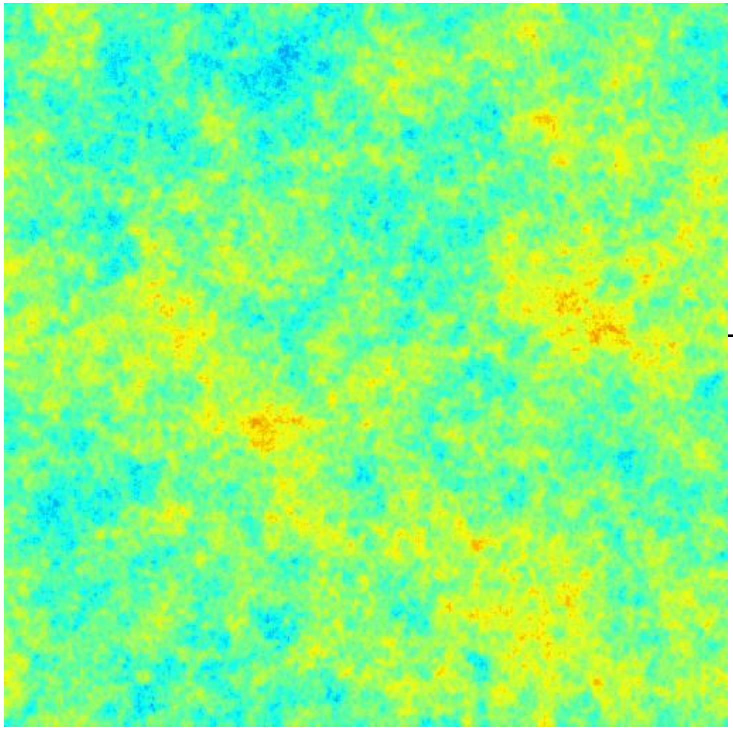
Observations: the microwave sky today

0^{th} order (uniform 2.726K) + 1^{st} order perturbations (anisotropies)

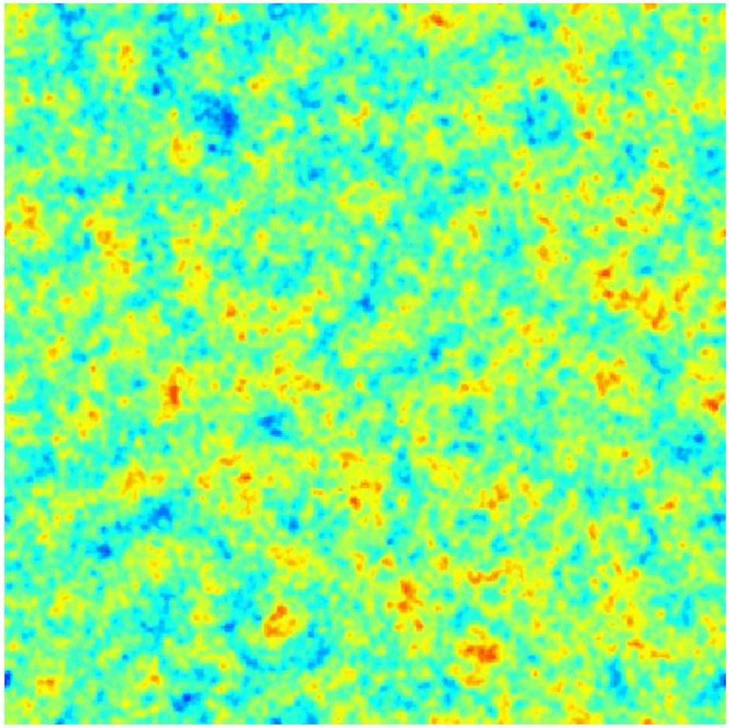
CMB temperature

0th order uniform temperature + 1st order perturbations:

Perturbations: End of inflation



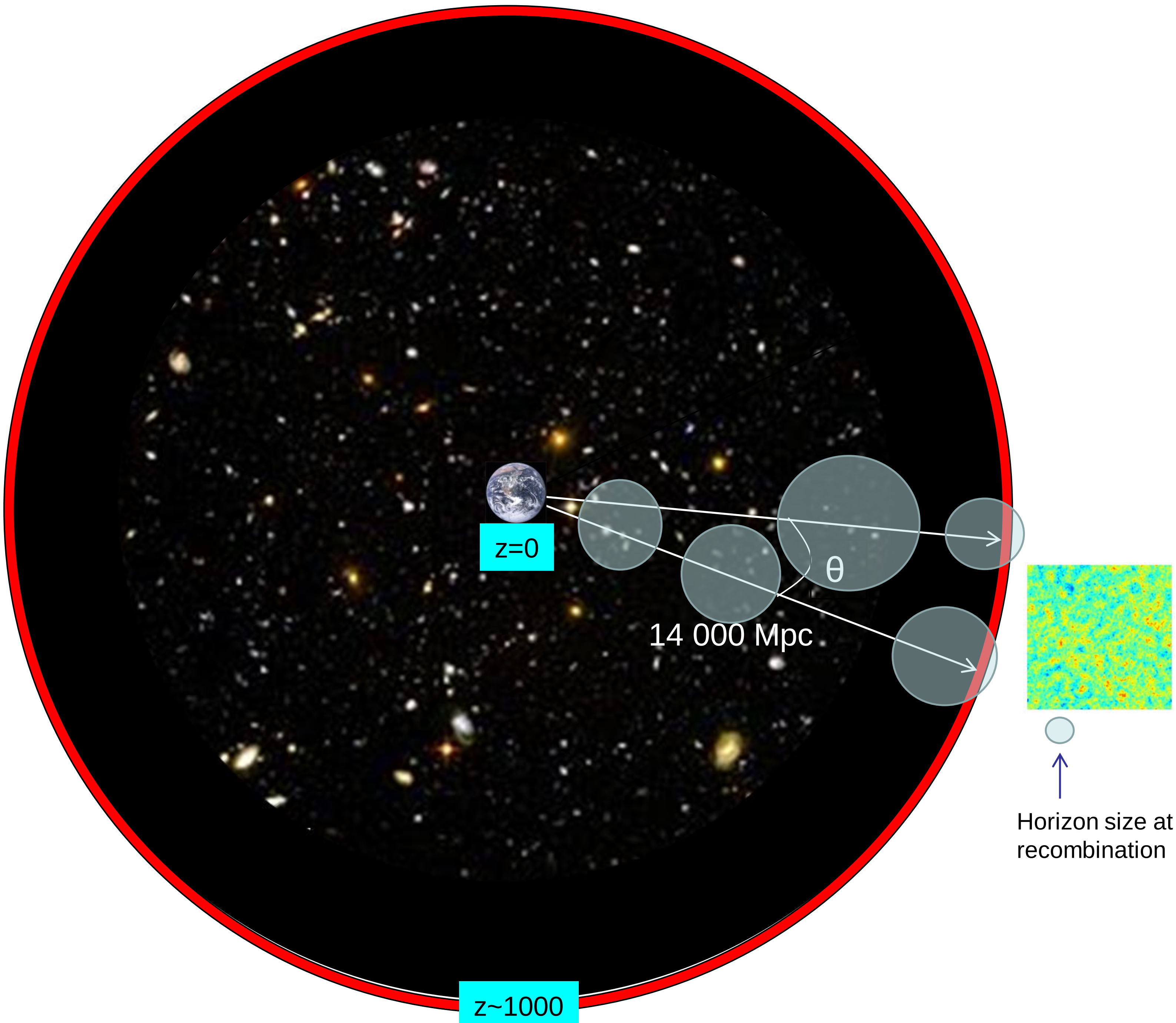
gravity+
pressure+
diffusion



Perturbations: Last scattering surface

Perturbations super-horizon

Sub-horizon acoustic oscillations
+ modes that are still super-horizon

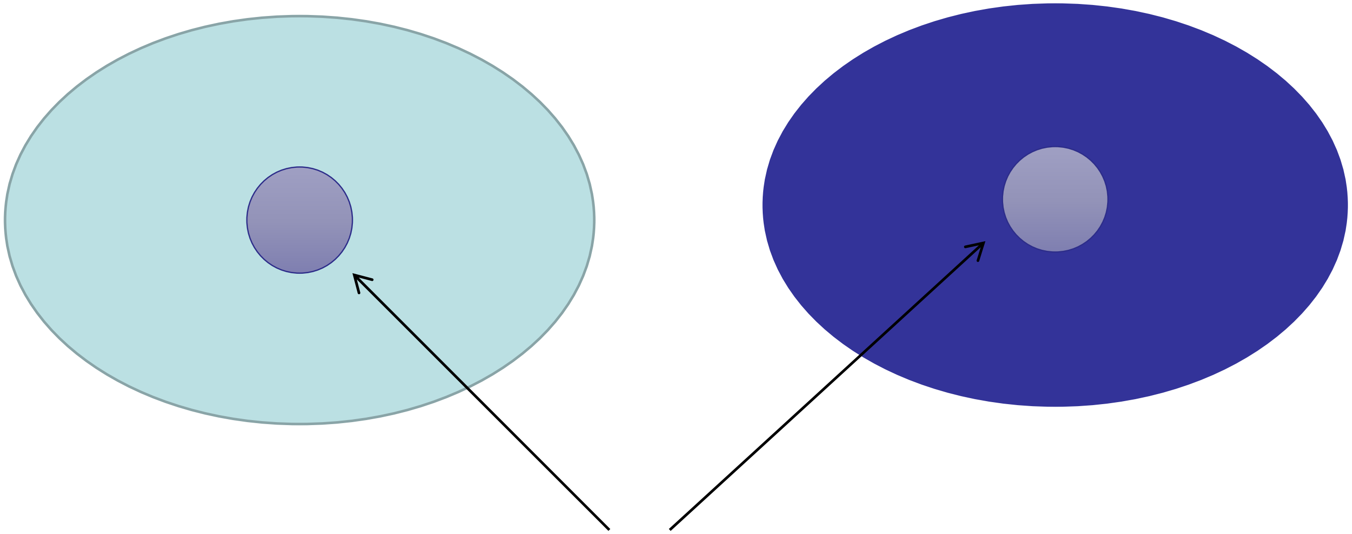


z=0
θ
14 000 Mpc
Horizon size at
recombination
z~1000

Effect of large-scale super-horizon perturbations?

Single-field inflation: only one degree of freedom, e.g. everything determined by local temperature (density) on super-horizon scales

Cannot locally observe super-horizon perturbations (to $O(\frac{k^2}{H^2})$)



Observers in different places on LSS will see statistically exactly the same thing
(at given fixed temperature/time from hot big bang)
- local physics is identical in Hubble patches that differ only by super-horizon modes

Universe recombines at same temperature everywhere; recombination is a constant temperature surface

BUT: a distant observer *will* see modulations due to the large modes <~ horizon size today
- can see and compare multiple different Hubble patches at recombination

- Linear modes cause anisotropic redshifting along the line of sight
  - 0^{th} order uniform last scattering surface modulated by 1^{st} order perturbations

⇒ generates linear CMB anisotropies on large scales, including ISW

- linear perturbations at last scattering are observed in perturbed universe:
  -1^{st} order small-scale perturbations are modulated by the effect 1^{st} order large-scale (and smaller-scale) modes

⇒ non-linear CMB anisotropies, mainly CMB lensing (2^{nd} order and higher)

For simplicity consider recombination to give sharp visibility –
CMB photons come from spherical shell about us at background time η_*

Need to use geodesic equation to account for line light propagation:

Us

LSS

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0$$

Affine parameter λ

4-momentum $p^\mu = \frac{dx^\mu}{d\lambda}$

(this defines choice of normalization of λ)

Use linear perturbation theory with $ds^2 = a(\eta)^2 \left[(1+2\Psi) d\eta^2 - (1-2\Phi) d\mathrm{x}^2 \right]$

- Conformal Newtonian Gauge (CNG) [scalar perturbations]
- Note signature convention different compared to CMB Theory lectures

Linear CMB anisotropies

Note: perturbations Φ and Ψ are functions of time and position

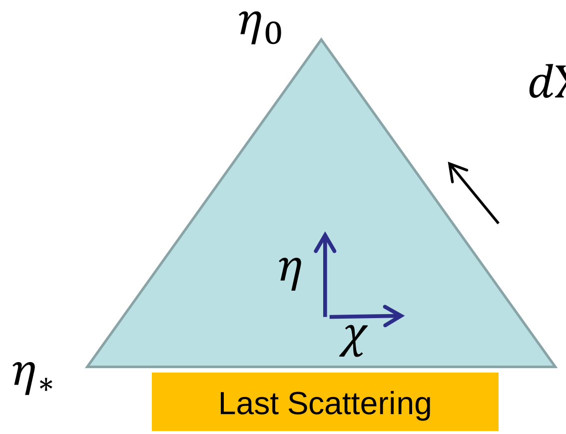
Zero component of geodesic equation in the Conformal Newtonian Gauge:

$$\frac{dp^0}{d\lambda} + \left(\frac{a'}{a} + \Psi'\right) p^0 p^0 + 2p^0 p^i \frac{\partial \Psi}{\partial x^i} + \left(\frac{a'}{a}(1 - 2\Psi - 2\Phi) - \Phi'\right) \delta_{ij} p^i p^j = 0$$

$$x^0 = \eta$$
$$x^r = \chi$$

Null geodesic:

$$d\eta = -d\chi$$



$$dX = \frac{\partial X}{\partial \eta}\, d\eta + \frac{\partial X}{\partial \chi}\, d\chi$$

$$\Rightarrow \frac{dX}{d\eta} = \frac{\partial X}{\partial \eta} - \frac{\partial X}{\partial \chi}$$

$$p^0 = E(1-\Psi)/a$$

$$\frac{d\eta}{d\lambda} = \frac{E}{a}(1-\Psi)$$

$$\Rightarrow$$

$$\frac{d(aE)}{d\eta} = aE\left(\frac{\partial\Psi}{\partial\chi} + \Phi'\right) = aE\left(-\frac{d\Psi}{d\eta} + \Psi' + \Phi'\right)$$

Integrate between time η and today (η_0), rearrange

$$E(\eta_0) = a(\eta)E(\eta)\left[1 + \Psi(\eta) - \Psi_0 + \int_\eta^{\eta_0} d\eta(\Psi' + \Phi')\right]$$

All photons redshift the same way, so $kT \sim E$.

Recombination fairly sharp at background time η_*: ~ *constant temperature* surface.
Also add Doppler effect:

$$\begin{aligned} T(\hat{\mathbf{n}}, \eta_0) &= (a_* + \delta a)T_*\left[1 + \Psi(\eta_*) - \Psi_0 + \hat{\mathbf{n}}\cdot(\mathbf{v}_o - \mathbf{v}) + \int_{\eta_*}^{\eta_0} d\eta(\Psi' + \Phi')\right] \\ &= T_0\left[1 + \frac{\delta a}{a_*} + \Psi(\eta_*) - \Psi_0 + \hat{\mathbf{n}}\cdot(\mathbf{v}_o - \mathbf{v}) + \int_{\eta_*}^{\eta_0} d\eta(\Psi' + \Phi')\right] \end{aligned}$$

$$\rho_\gamma \propto T^4 \propto a^4$$

$$\Rightarrow \quad \frac{\Delta T_0}{T}(\hat{\mathbf{n}}) = \frac{\Delta_\gamma(\eta_*)}{4} + \underbrace{\Psi(\eta_*) - \Psi_0}_{\text{Sachs-Wolfe}} + \underbrace{\hat{\mathbf{n}} \cdot (\mathbf{v}_o - \mathbf{v})}_{\text{Doppler}} + \underbrace{\int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi')}_{\text{ISW}}$$

Temperature
perturbation at
recombination
(Newtonian Gauge)

Alternative $$\frac{dX}{d\eta} = \frac{\partial X}{\partial \eta} - \frac{\partial X}{\partial \chi}$$

$$a_A E_A = a(\eta)E(\eta)\left[1 + \Psi(\eta) - \Psi_A + \int_\eta^{\eta_A} \mathrm{d}\eta \partial_\eta (\Psi + \Phi)\right]$$

$$= a(\eta)E(\eta)\left[1 - \Phi(\eta) + \Phi_A + \int_\eta^{\eta_A} \mathrm{d}\eta \partial\chi (\Psi + \Phi)\right]$$

$$\frac{\Delta T_{\text{obs}}}{T}(\hat{\mathbf{n}}) := \frac{\Delta_\gamma}{4} - \Phi + \hat{\mathbf{n}} \cdot (\mathbf{v}_A - \mathbf{v}) + \int_\eta^{\eta_A} \mathrm{d}\eta\, \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}(\Psi + \Phi)$$

$$= \zeta_\gamma + \hat{\mathbf{n}} \cdot (\mathbf{v}_A - \mathbf{v}) + \int_\eta^{\eta_A} \mathrm{d}\eta\, \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}(\Psi + \Phi)$$

Gauge-invariant 3-curvature on constant temperature hypersurfaces $\equiv \delta N$ expansion in flat gauge

Writing Sachs-Wolfe source terms + ISW is just a convenient choice:
physics is the gradual anisotropic redshifting of photons along the line of sight

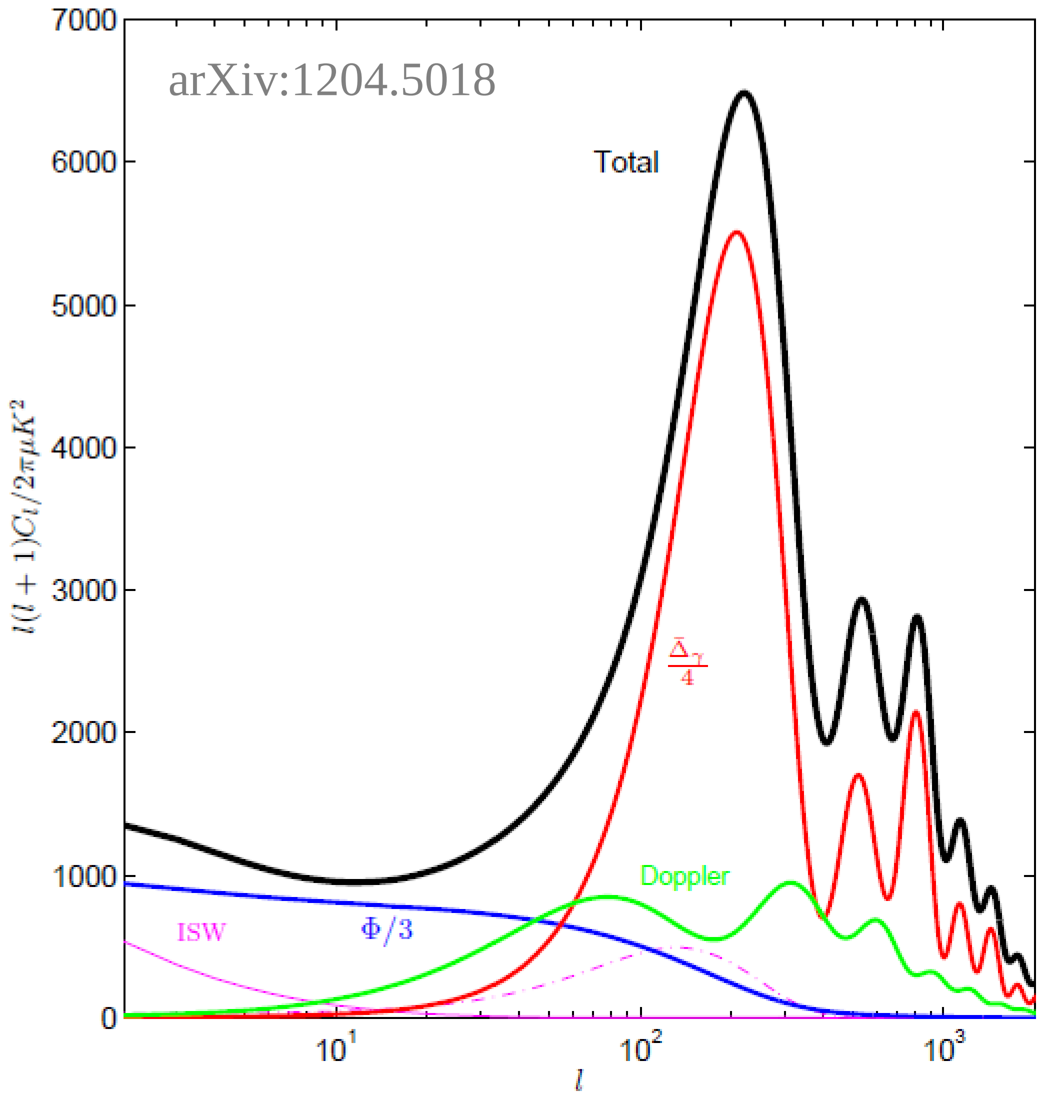
How do potentials evolve, Φ', Ψ'? In Newtonian limit $\Phi = \Psi \propto GM/r$

- Universe is expanding, physical size of perturbation $\propto a$, so $r \propto a$
- Density perturbations are growing, in matter domination $M \propto a^3 \delta\rho \propto \delta\rho/\rho \propto a$

In matter domination $\Phi = \Psi = const$: ISW term vanishes – contributions separate in time

Early ISW: changing potentials going from radiation to matter domination

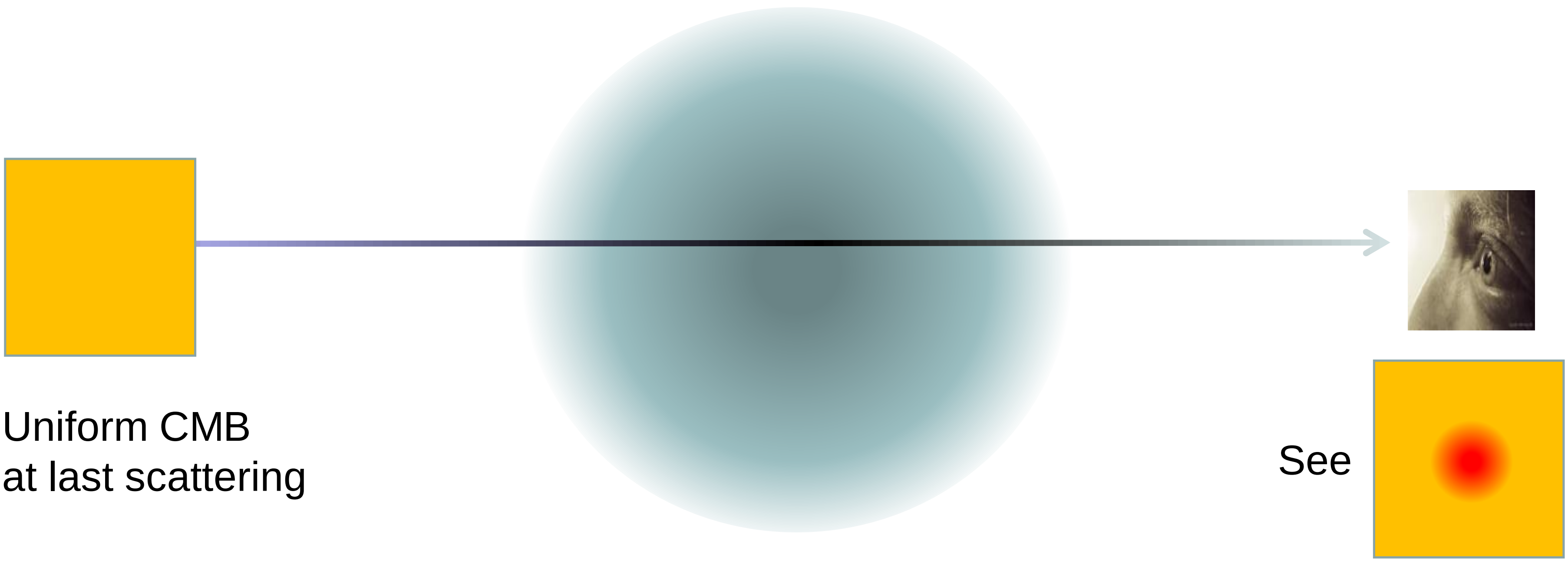
Late ISW: changing potentials as dark energy starts to be important



arXiv:1204.5018
Total
$\frac{\bar{\Delta}_\gamma}{4}$
Doppler
ISW
$\Phi/3$
$l(l+1)C_l/2\pi\mu K^2$
l
7000
6000
5000
4000
3000
2000
1000
0
10^1
10^2
10^3

Radiation negligible $\Rightarrow \Psi = \Phi$

$$\Delta T_{\rm ISW}(\hat{\mathbf{n}}) = 2\int_0^{\chi_*} \mathrm{d}\chi\, \dot{\Psi}(\chi\hat{\mathbf{n}}; \eta_0 - \chi)$$



Late time: dark energy slows growth of structure (expansion faster than growth)
– potentials decay with time

Overdensity: positive ISW (net blueshift, deeper potential falling in than climbing out)

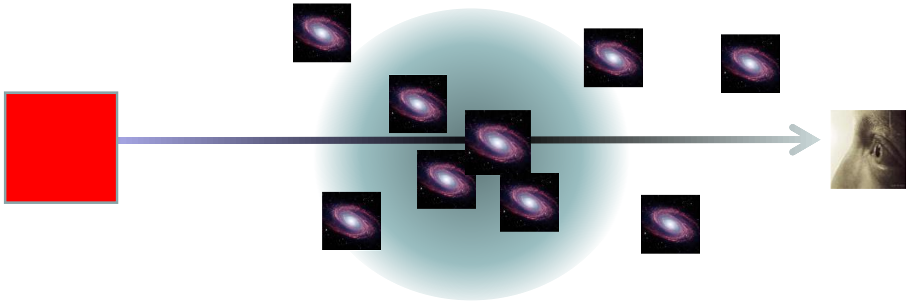
Underdensity: negative ISW (net redshift)

Why the interest in ISW? Probes late time evolution: constrain dark energy

Problem: CMB anisotropy is ISW+Sachs-Wolfe+Doppler+…: cannot easily isolate

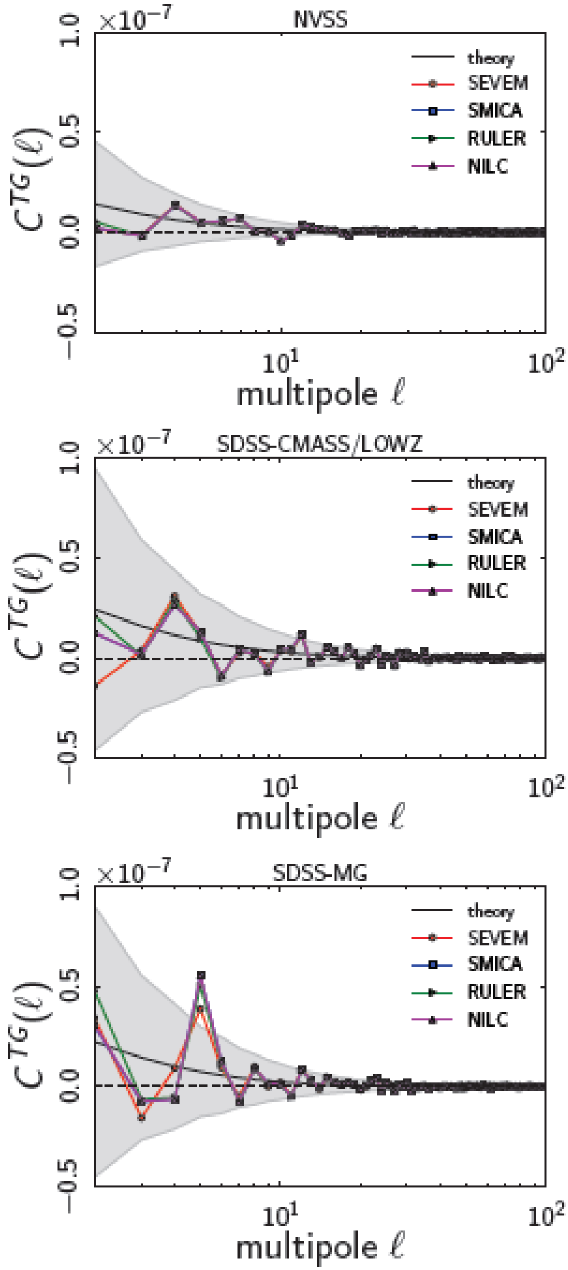
Partial solution: correlate CMB with another probe of the large-scale structure

Big overdensity ⇒ 1. Positive ISW signal, 2. Higher density of galaxies



Positive correlation $\langle T\, n_g \rangle > 0$

Planck 2013 results. XIX. The integrated Sachs-Wolfe effect



Detected, consistent with cosmological constant.
But still limited in principle by cosmic variance to $\sim 7\sigma$

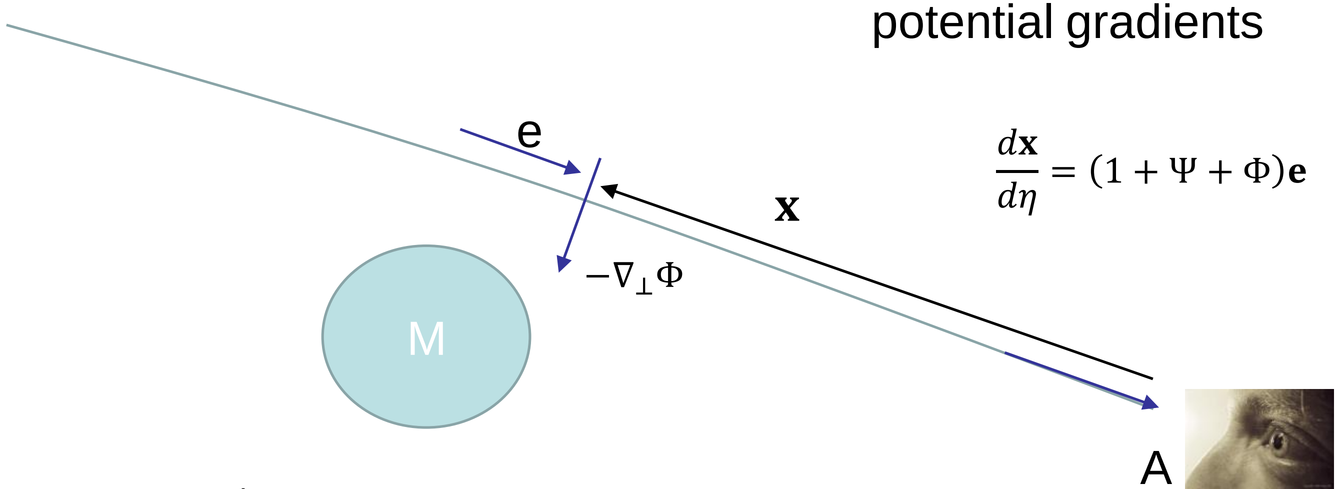
LSS data	$\hat{\xi}_a^{xy}$	C-R		NILC		SEVEM		SMICA	
NVSS	CAPS	0.86 ± 0.33	2.6	0.91 ± 0.33	2.8	0.90 ± 0.33	2.7	0.91 ± 0.33	2.7
	CCF	0.80 ± 0.33	2.4	0.84 ± 0.33	2.5	0.83 ± 0.33	2.5	0.84 ± 0.33	2.5
	SMHWcov	0.89 ± 0.34	2.6	0.93 ± 0.34	2.8	0.89 ± 0.34	2.6	0.92 ± 0.34	2.7
SDSS-CMASS/LOWZ	CAPS	0.98 ± 0.52	1.9	1.09 ± 0.52	2.1	1.06 ± 0.52	2.0	1.09 ± 0.52	2.1
	CCF	0.81 ± 0.52	1.6	0.91 ± 0.52	1.8	0.89 ± 0.52	1.7	0.90 ± 0.52	1.7
	SMHWcov	0.80 ± 0.53	1.5	0.89 ± 0.53	1.9	0.87 ± 0.53	1.6	0.88 ± 0.53	1.7
SDSS-MG	CAPS	1.31 ± 0.57	2.3	1.43 ± 0.57	2.5	1.35 ± 0.57	2.4	1.42 ± 0.57	2.5
	CCF	1.00 ± 0.57	1.8	1.11 ± 0.57	2.0	1.10 ± 0.57	1.9	1.10 ± 0.57	1.9
	SMHWcov	1.03 ± 0.59	1.8	1.18 ± 0.59	2.0	1.15 ± 0.59	2.0	1.17 ± 0.59	2.0
all	CAPS	0.84 ± 0.31	2.7	0.91 ± 0.31	2.9	0.88 ± 0.31	2.0	0.90 ± 0.31	2.9
	CCF	0.77 ± 0.31	2.5	0.83 ± 0.31	2.7	0.82 ± 0.31	2.6	0.82 ± 0.31	2.7
	SMHWcov	0.86 ± 0.32	2.7	0.92 ± 0.32	2.9	0.89 ± 0.32	2.8	0.91 ± 0.32	2.9

Spatial components of the geodesic equation?

$$\frac{d\mathrm{e}}{d\eta} = -\nabla_{\perp}(\Phi + \Psi)$$

Where $\mathbf{e}$ is the spatial photon propagation direction
- deflection due to transverse potential gradients

$$\frac{d\mathbf{x}}{d\eta} = (1 + \Psi + \Phi)\mathbf{e} \qquad \frac{d\eta}{d\lambda} = \frac{E}{a}(1 - \Psi)$$



Full solution for 0th+1st order photon path

$$\mathrm{x}(\hat{\mathrm{n}};\eta) = \underbrace{-\mathrm{e}_A(\eta_A - \eta)}_{\text{FRW background solution}} + \underbrace{\mathrm{e}_A \int_{\eta_A}^{\eta} (\Phi + \Psi)\, \mathrm{d}\eta'}_{\text{Time delay}} \underbrace{- \int_{\eta_A}^{\eta} (\eta - \eta')\nabla_{\perp}(\Phi + \Psi)\mathrm{d}\eta'}_{\text{Lensing}}$$

Zeroth-order CMB

- CMB uniform blackbody at ~2.7 K
  (+dipole due to local motion)

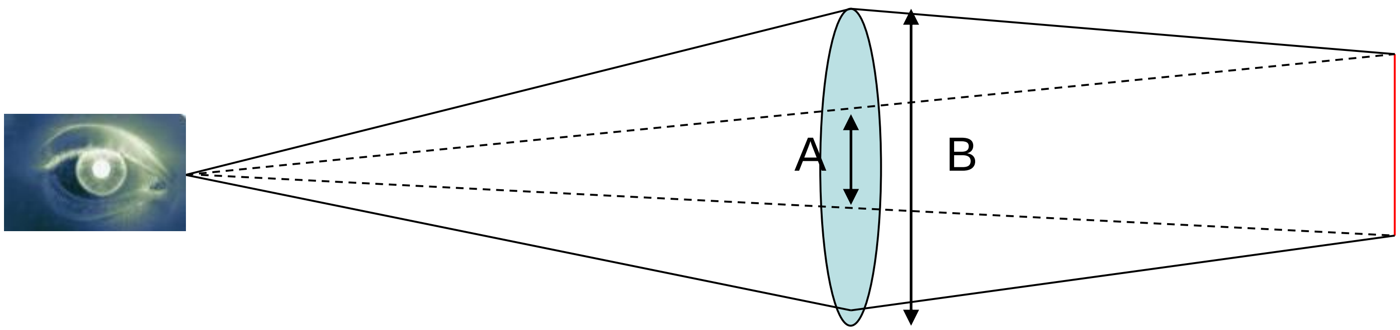
1^{st} order effects

- Linear perturbations at last scattering, zeroth-order light propagation; zeroth-order last scattering, first order redshifting during propagation (ISW)
  - usual unlensed CMB anisotropy calculation

- First order time delay, uniform CMB
  - last scattering displaced, but temperature at recombination the same
  - no observable effect

1^{st} order effects contd.

- First order CMB lensing: zeroth-order last scattering (uniform CMB ~ 2.7K), first order transverse displacement in light propagation



$$\frac{\text{Number of photons before lensing}}{\text{Number of photons after lensing}} = \frac{A^2}{B^2} = \frac{\text{Solid angle before lensing}}{\text{Solid angle after lensing}}$$

<u>Conservation of surface brightness</u>: number of photons per solid angle unchanged

uniform CMB lenses to uniform CMB – so no observable effect

2^{nd} order effects

- Second order perturbations at last scattering, zeroth order light propagation
  -tiny $\sim(10^{-5})^2$ corrections to linear unlensed CMB result

- First order last scattering ($\sim 10^{-5}$ anisotropies), first order transverse light displacement
  - this is what we call CMB lensing

- First order last scattering, first order time delay
  - delay ~1MPc, small compared to thickness of last scattering
  - coherent over large scales: very small observable effect

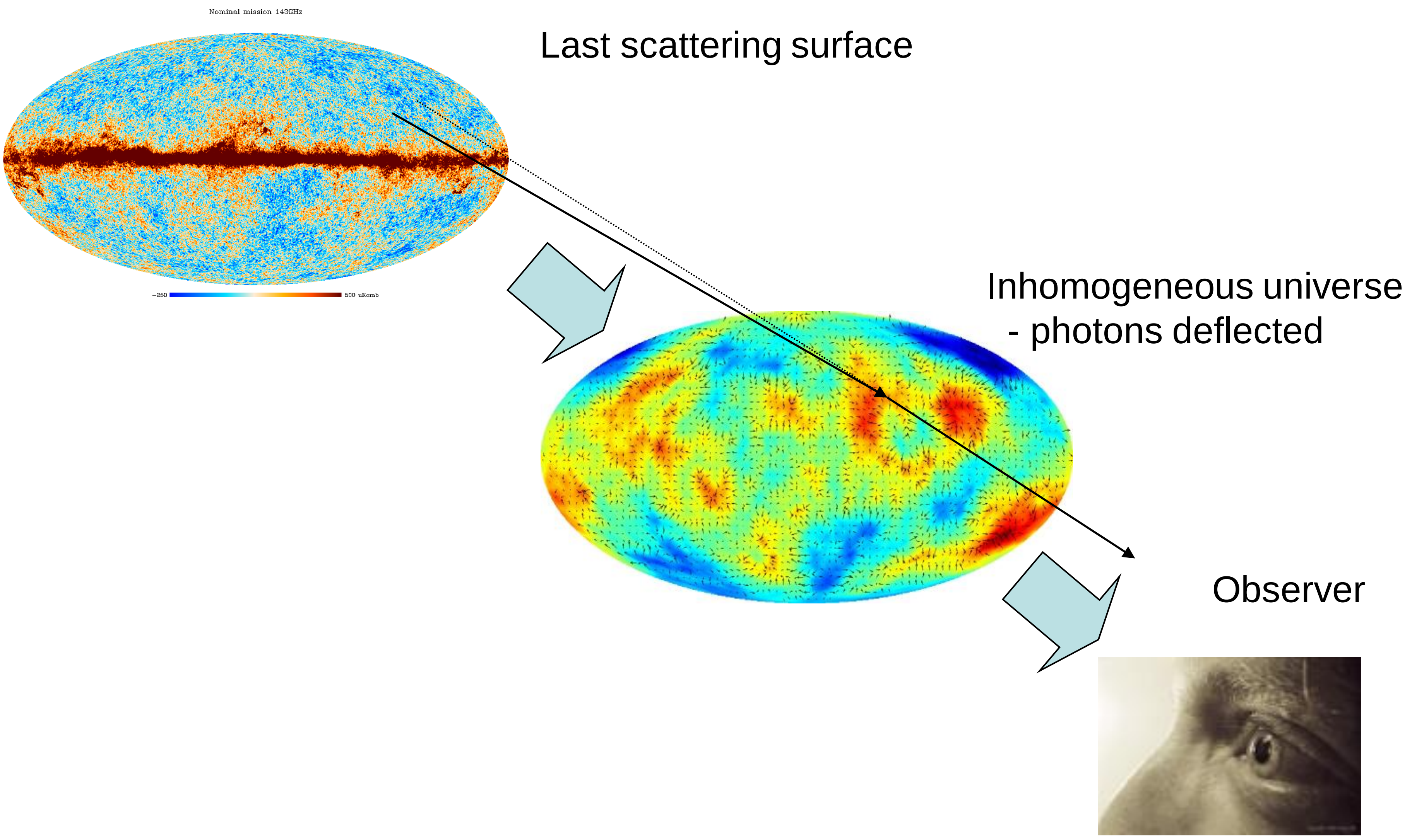
Hu, Cooray: astro-ph/0008001

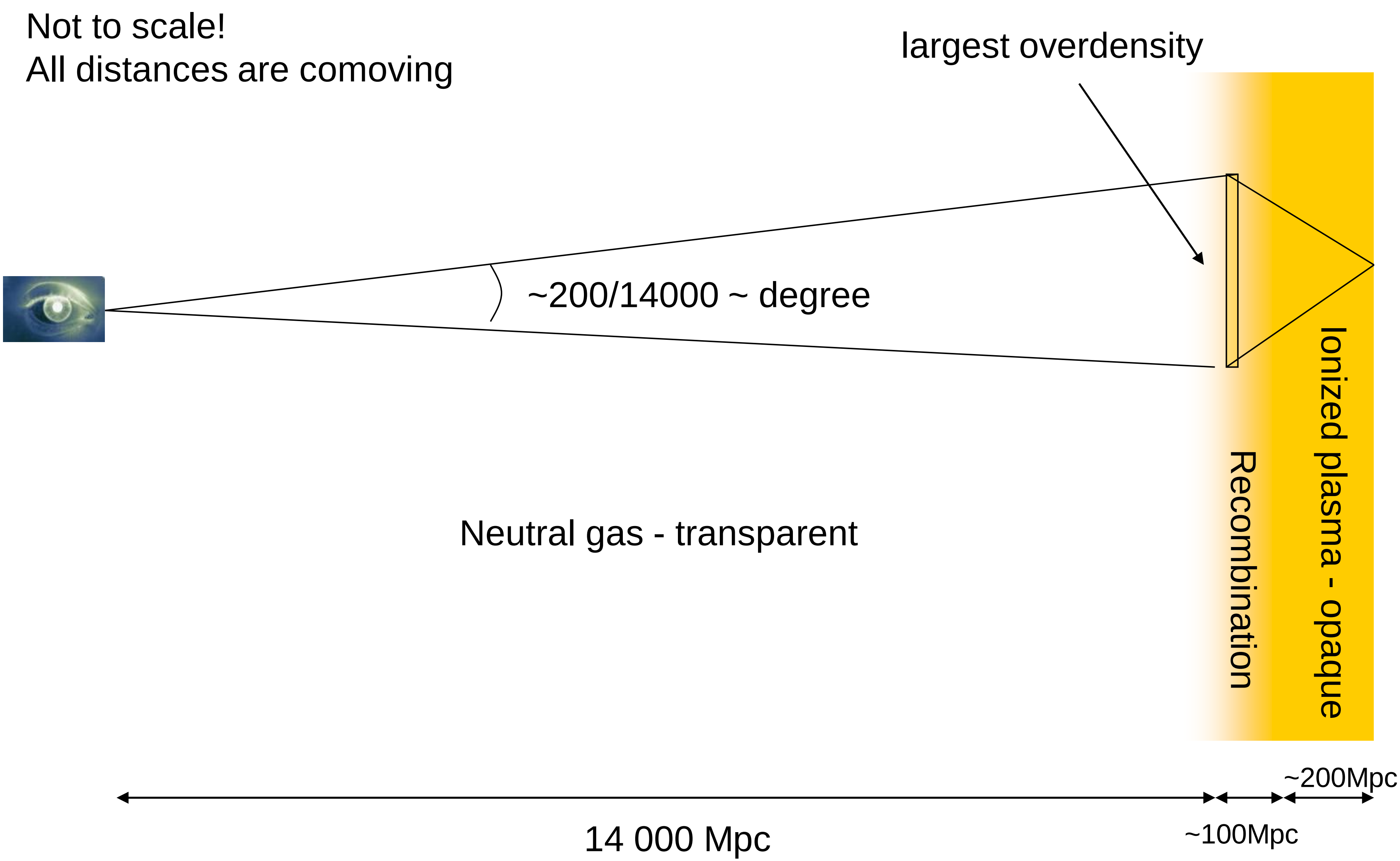
- First order last scattering, first order anisotropic expansion
  $\sim(10^{-5})^{2:}$ small but non-zero contribution to large-scale bispectrum
  [equivalent to mapping from physical to comoving $\mathbf{x}$ - the Maldacena consistency relation bispectrum on the CMB]

- First order last scattering, first order anisotropic redshifting
  $\sim(10^{-5})^2$: gives non-zero but very small contribution to large-scale bispectrum

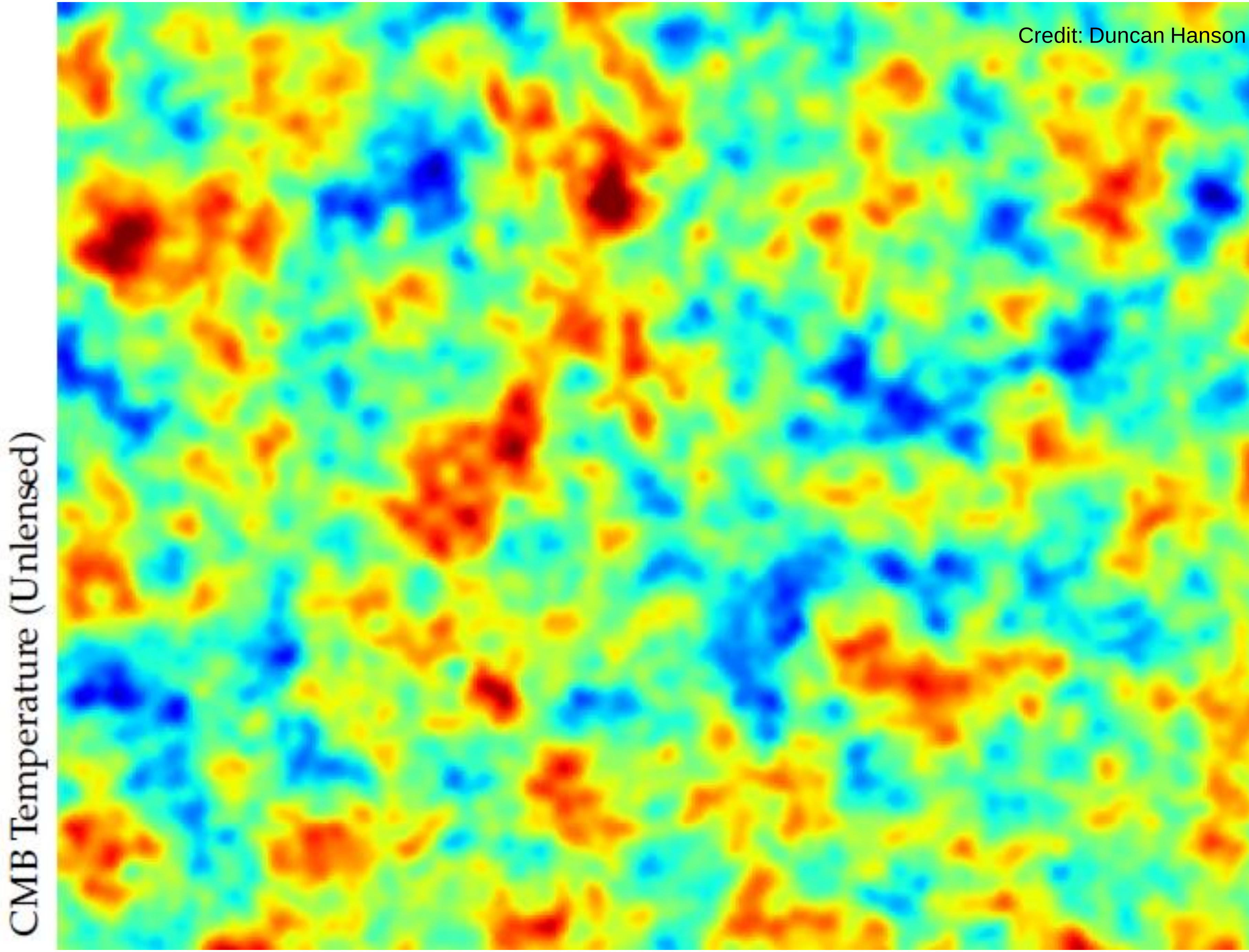
- Others
  e.g. Rees-Sciama: second (+ higher) order redshifting
  SZ: second (+higher) order scattering, etc….

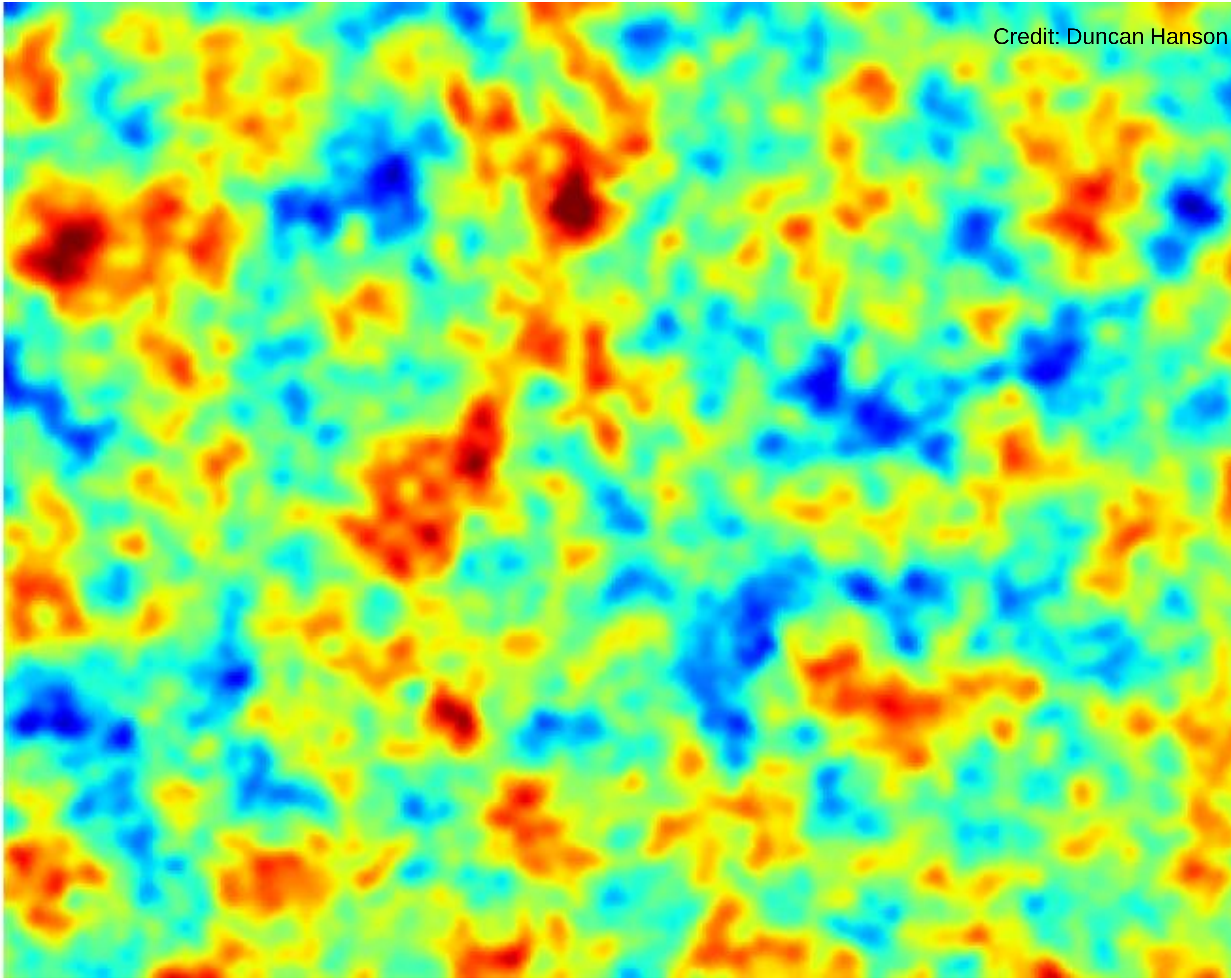
Weak lensing of the CMB perturbations





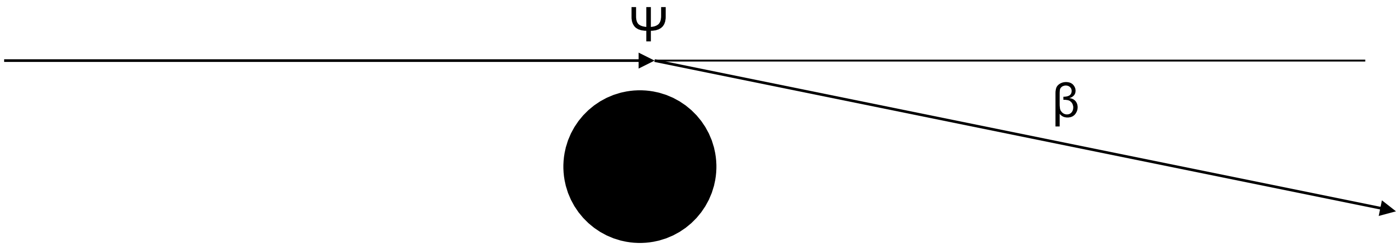
Good approximation: CMB is single source plane at ~14 000 Mpc


Credit: Duncan Hanson
CMB Temperature (Unlensed)



CMB Temperature (Lensed)

CMB lensing order of magnitudes



Newtonian argument: $\beta = 2\,\Psi$

General Relativity: $\beta = 4\,\Psi$

$(\beta \ll 1)$

Potentials linear and approx Gaussian: $\Psi \sim 2 \times 10^{-5}$

$\beta \sim 10^{-4}$

How big are the lenses/how many of them?

Matter Power Spectrum

(in comoving gauge)

$$\Delta = \delta\rho/\rho \qquad \langle \Delta(\mathbf{k},t)\Delta(\mathbf{k}',t)\rangle = \frac{2\pi^2}{k^3}\mathcal{P}(k,t)\delta(\mathbf{k}+\mathbf{k}')$$

Large scales, $k \ll aH_{eq}$: Use Poisson equation $\bar{\Delta} = -(2/3)k^2\Phi/\mathcal{H}^2$

$$\mathcal{P}_{\bar{\Delta}}(\eta) \sim \frac{4}{9}\frac{k^4}{\mathcal{H}^4}\mathcal{P}_\Phi = \frac{4}{25}\frac{k^4}{\mathcal{H}^4}\mathcal{P}_\mathcal{R}$$

Small scales, $k \gg aH_{eq}$:

(+ matter domination)

$$\mathcal{P}_{\bar{\Delta}} \sim \left(\frac{9\pi^2}{16}\right)^2 \frac{a^2}{a_{eq}^2}\left[1 + 2\ln\left(\frac{4k\eta_{eq}}{\pi 3\sqrt{3}}\right)\right]^2 \mathcal{P}_\mathcal{R}$$

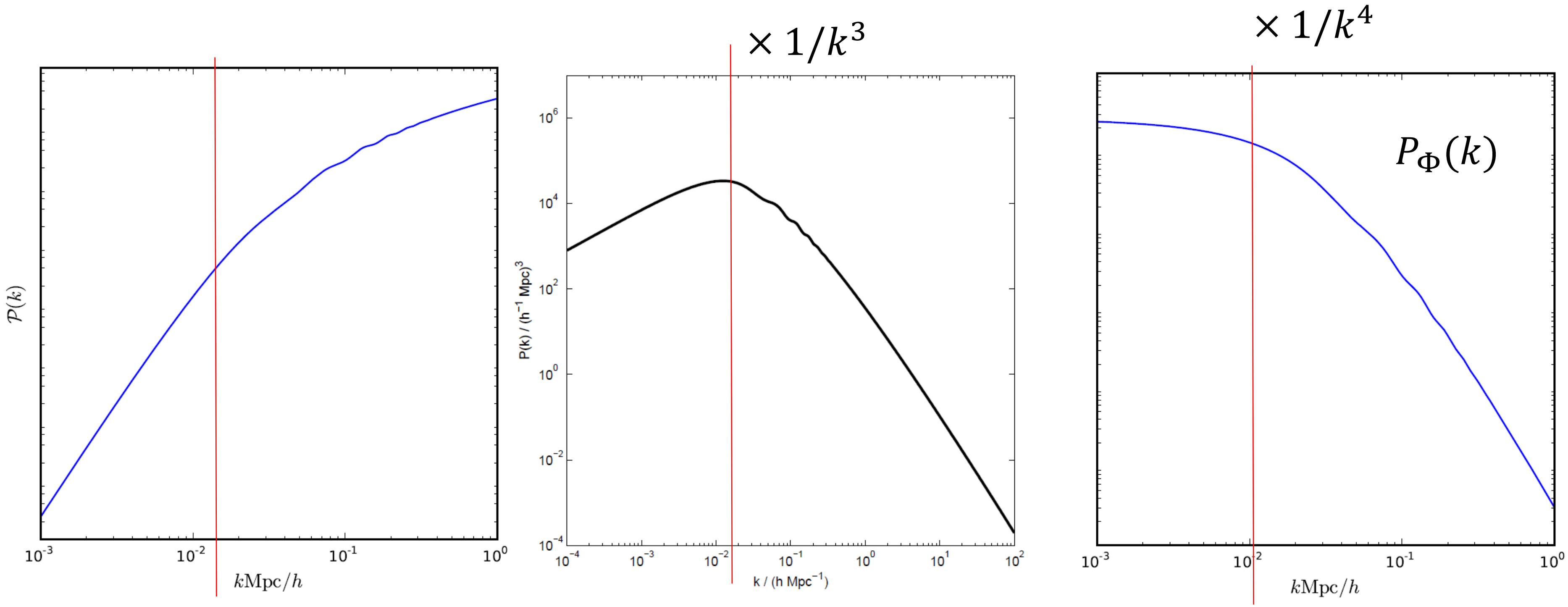
Structure growth in matter domination $\frac{\delta\rho}{\rho} = \Delta \propto a$

Growth during radiation domination.

Photon pressure stops growth: $\Phi \to 0$ due to expansion
⇒ no gravitational driving force, no acceleration
⇒ dark matter velocities redshift $\propto 1/a$
Integrate $v \propto 1/a$ to get density ⇒ $\ln(\eta)$ growth

For more details see notes at: http://cosmologist.info/teaching/EU/

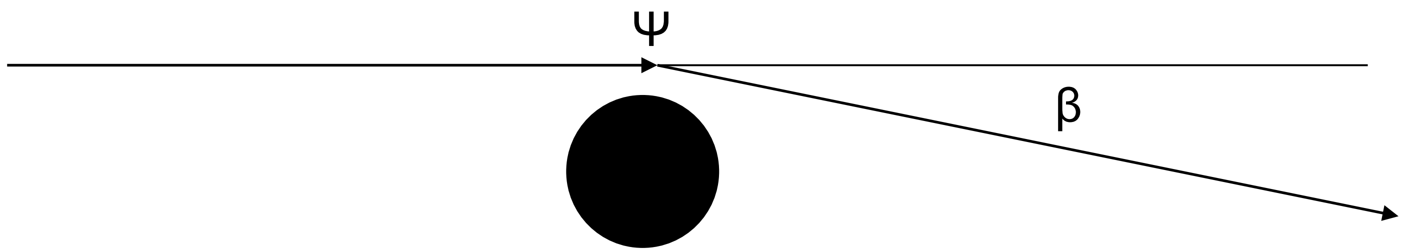
Linear Matter Power Spectrum



Turnover in matter power spectrum at $k \sim 0.01 - 0.02$
(set by horizon size at matter-radiation equality)

More lenses ⇒ more lensing ⇒ most effect for small lenses for more along line of sight
Smallest lenses where potential has not decayed away $\sim 300\mathrm{Mpc}$

CMB lensing order of magnitudes



Newtonian argument: $\beta = 2\,\Psi$

General Relativity: $\beta = 4\,\Psi$

($\beta \ll 1$)

Potentials linear and approx Gaussian: $\Psi \sim 2 \times 10^{-5}$

$\beta \sim 10^{-4}$

Characteristic size from peak of matter power spectrum ~ 300Mpc

Comoving distance to last scattering surface ~ 14000 Mpc

pass through ~50 lumps

assume uncorrelated

total deflection ~ $50^{1/2} \times 10^{-4}$

~ 2 arcminutes

(neglects angular factors, correlation, etc.)

Why lensing is important

Relatively large $O(10^{-3})$ *not* $O(10^{-5})$ – GR lensing factor, many lenses along line of sight
[*NOT* because of growth of matter density perturbations, potentials are constant or decaying!]

- 2arcmin deflections: $l \sim 3000$
  - On small scales CMB is very smooth so lensing dominates the linear signal at high l

- Deflection angles coherent over 300/(14000/2) ~ 2°
  - comparable to CMB scales
  - expect 2arcmin/60arcmin ~ 3% effect on main CMB acoustic peaks

- Non-linear: observed CMB is non-Gaussian
  - more information
  - potential confusion with primordial non-Gaussian signals

- Does not preserve E/B decomposition of polarization: e.g. $E \rightarrow B$
  - Confusion for primordial B modes ("r-modes")
  - No primordial B ⇒ B modes clean probe of lensing

Deflection angle α, shear γ_i and convergence κ

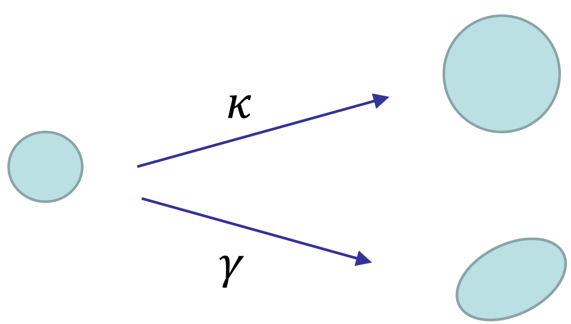
- Often switch between equivalent alternative descriptions

$$T(\hat{\mathbf{n}}) = T(\hat{\mathbf{n}}') = T(\hat{\mathbf{n}} + \boldsymbol{\alpha})$$

$$A_{ij} \equiv \delta_{ij} + \frac{\partial}{\partial \theta_i}\alpha_j = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2+\omega \\ -\gamma_2-\omega & 1-\kappa+\gamma_1 \end{pmatrix}$$

Rotation $\omega = 0$ from scalar perturbations in linear perturbation theory (because deflections from gradient of a potential)

Small source $\quad I(\hat{\mathbf{n}} + \delta\xi) \quad \rightarrow \quad I(\hat{\mathbf{n}}' + \mathbf{A}\delta\xi)$



Convergence $\quad \kappa = -\frac{1}{2}\nabla \cdot \boldsymbol{\alpha}$

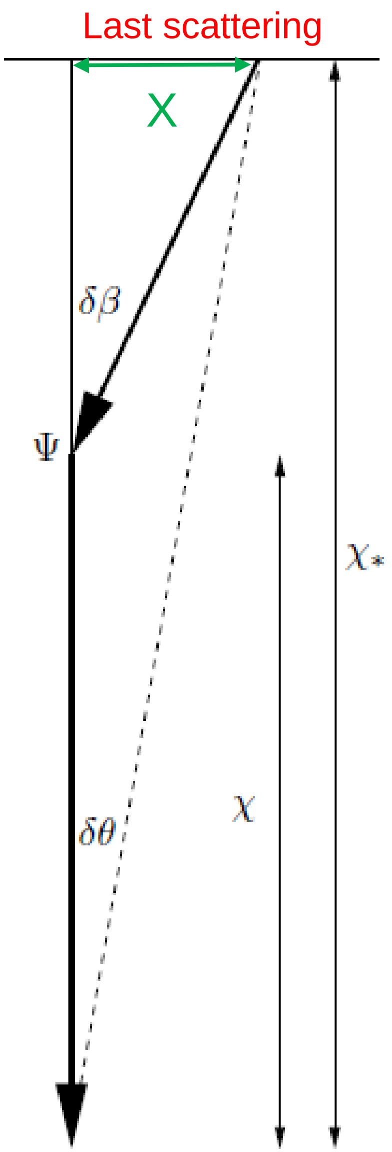
Shear

Calculating the deflection angles

$$\frac{de}{d\eta} = -\nabla_\perp \underbrace{(\Phi + \Psi)}$$

Weyl Potential:
$\Psi_W \equiv (\Phi + \Psi)/2$
determines scalar
part of the Weyl tensor
Conformally invariant

$$\delta\beta = -2\delta\chi\nabla_\perp\Psi$$



FRW background: comoving angular diameter distance

$$f_K(\chi) = \begin{cases} K^{-1/2}\sin(K^{1/2}\chi) & \text{for } K > 0, \text{ closed}, \\ \chi & \text{for } K = 0, \text{ flat}, \\ |K|^{-1/2}\sinh(|K|^{1/2}\chi) & \text{for } K < 0, \text{ open}. \end{cases}$$

Write X, in two ways $\quad f_K(\chi_* - \chi)\delta\beta = f_K(\chi_*)\delta\theta$

$$\Rightarrow \quad \delta\theta_\chi = \frac{f_K(\chi_* - \chi)\delta\beta}{f_K(\chi_*)} = -\frac{f_K(\chi_* - \chi)}{f_K(\chi_*)} 2\delta\chi\nabla_\perp\Psi$$

Observed deflection

Lensed temperature depends on deflection angle

$$\tilde{T}(\hat{\mathbf{n}}) = T(\hat{\mathbf{n}}') = T(\hat{\mathbf{n}} + \boldsymbol{\alpha})$$

Newtonian potential

$$\boldsymbol{\alpha} = \delta\theta = -2 \int_0^{\chi^*} \mathrm{d}\chi \frac{f_K(\chi^* - \chi)}{f_K(\chi^*)} \nabla_\perp \Psi(\chi\hat{\mathbf{n}}; \eta_0 - \chi)$$

co-moving distance to last scattering

See lensing review for more rigorous spherical derivation

Lensing Potential

Deflection angle on sky given in terms of angular gradient of lensing potential $\boldsymbol{\alpha} = \nabla\psi$

$$\boldsymbol{\nabla}_\perp \Psi = (\nabla_{\hat{\mathbf{n}}} \Psi)/f_K(\chi)$$

$$\Rightarrow \quad \psi(\hat{\mathbf{n}}) = -2 \int_0^{\chi_*} \mathrm{d}\chi \, \Psi(\chi\hat{\mathbf{n}}; \eta_0 - \chi) \frac{f_K(\chi^* - \chi)}{f_K(\chi^*) f_K(\chi)}$$

$$\tilde{X}(\mathbf{n}) = X(\mathbf{n}') = X(\mathbf{n} + \nabla\psi(\mathbf{n}))$$

Comparison with galaxy lensing

- **Single source plane at known distance**
  (given cosmological parameters)

- **Statistics of sources on source plane well understood**
  - can calculate power spectrum; Gaussian linear perturbations
  - magnification and shear information equally useful - usually discuss in terms of deflection angle;
  - magnification analysis of galaxies much more difficult

- **Hot and cold spots are large, smooth on small scales**
  - 'strong' and 'weak' lensing can be treated the same way: infinite magnification of smooth surface is still a smooth surface

- **Source plane very distant, large linear lenses**
  - lensing by under- and over-densities;

- **Full sky observations**
  - may need to account for spherical geometry for accurate results

Power spectrum of the lensing potential

Expand Newtonian potential in 3D harmonics

$$\Psi(\mathbf{x};\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\,\Psi(\mathbf{k};\eta)e^{i\mathbf{k}\cdot\mathbf{x}},$$

with power spectrum

$$\langle \Psi(\mathbf{k};\eta)\Psi^*(\mathbf{k}';\eta')\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_\Psi(k;\eta,\eta')\delta(\mathbf{k}-\mathbf{k}').$$

Angular correlation function of lensing potential:

$$\langle \psi(\hat{\mathbf{n}})\psi(\hat{\mathbf{n}}')\rangle =$$

$$4\int_0^{\chi_*}\mathrm{d}\chi\int_0^{\chi_*}\mathrm{d}\chi'\left(\frac{\chi_*-\chi}{\chi_*\chi}\right)\left(\frac{\chi_*-\chi'}{\chi_*\chi'}\right)\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{2\pi^2}{k^3}\mathcal{P}_\Psi(k;\eta,\eta')e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{x}'}$$

Use $e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(k\chi) Y^*_{lm}(\hat{\mathbf{n}}) Y_{lm}(\hat{\mathbf{k}})$ j_l are spherical Bessel functions

Orthogonality of spherical harmonics (integral over k) then gives

$$\langle \psi(\hat{\mathbf{n}})\psi(\hat{\mathbf{n}}')\rangle = 16\pi \sum_{ll'mm'} \int_0^{\chi_*} \mathrm{d}\chi \int_0^{\chi_*} \mathrm{d}\chi' \left(\frac{\chi_* - \chi}{\chi_*\chi}\right)\left(\frac{\chi_* - \chi'}{\chi_*\chi'}\right) \times \int \frac{\mathrm{d}k}{k} j_l(k\chi) j_{l'}(k\chi') \mathcal{P}_\Psi(k;\eta,\eta') Y_{lm}(\hat{\mathbf{n}}) Y^*_{l'm'}(\hat{\mathbf{n}}')\delta_{ll'}\delta_{mm'}$$

Then take spherical transform using

$$\psi(\hat{\mathbf{n}}) = \sum_{lm} \psi_{lm} Y_{lm}(\hat{\mathbf{n}}), \qquad \langle \psi_{lm}\psi^*_{l'm'}\rangle = \delta_{ll'}\delta_{mm'} C_l^\psi.$$

Gives final general result

$$C_l^\psi = 16\pi \int \frac{\mathrm{d}k}{k} \int_0^{\chi_*} \mathrm{d}\chi \int_0^{\chi_*} \mathrm{d}\chi' \, \mathcal{P}_\Psi(k;\eta_0 - \chi, \eta_0 - \chi') j_l(k\chi) j_l(k\chi') \left(\frac{\chi_* - \chi}{\chi_*\chi}\right)\left(\frac{\chi_* - \chi'}{\chi_*\chi'}\right)$$

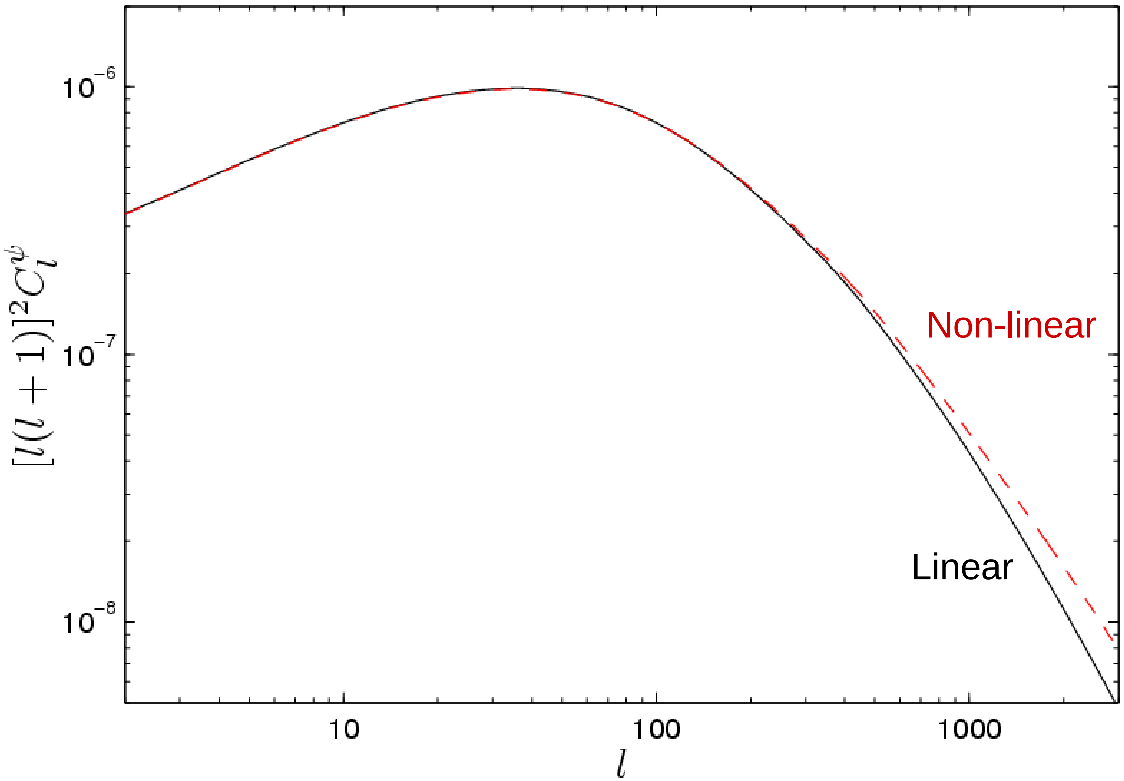
Deflection angle power spectrum

On small scales
(Limber approx, $k\chi \sim l$)

$$C_l^{\psi} \approx \frac{8\pi^2}{l^3} \int_0^{\chi_*} \chi \mathrm{d}\chi \, \mathcal{P}_\Psi(l/\chi; \eta_0 - \chi) \left(\frac{\chi_* - \chi}{\chi_* \chi} \right)^2$$

(better: $l \to l + 1/2$)

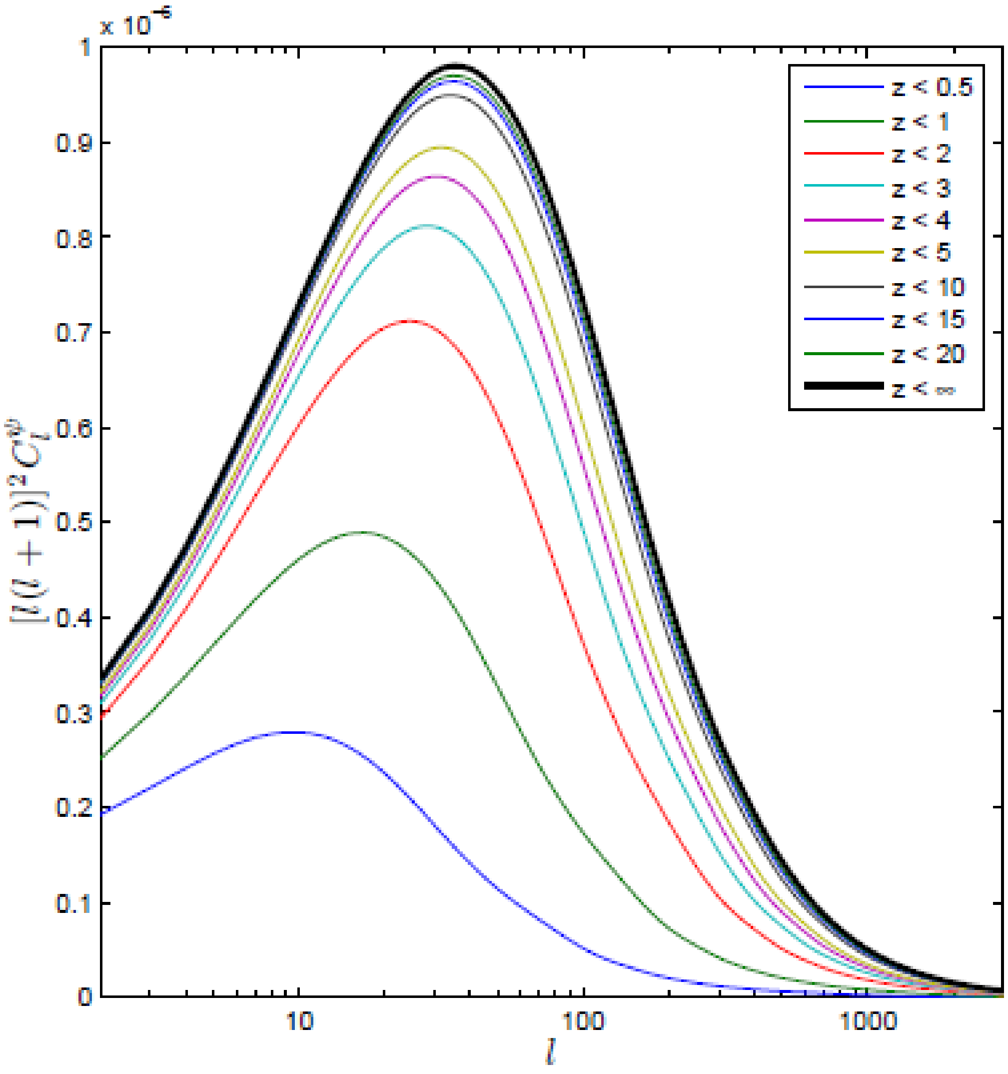
Deflection angle power ~ $l(l+1)C_l^{\psi}$



Deflections O(10^{-3}), but coherent on degree scales → important!

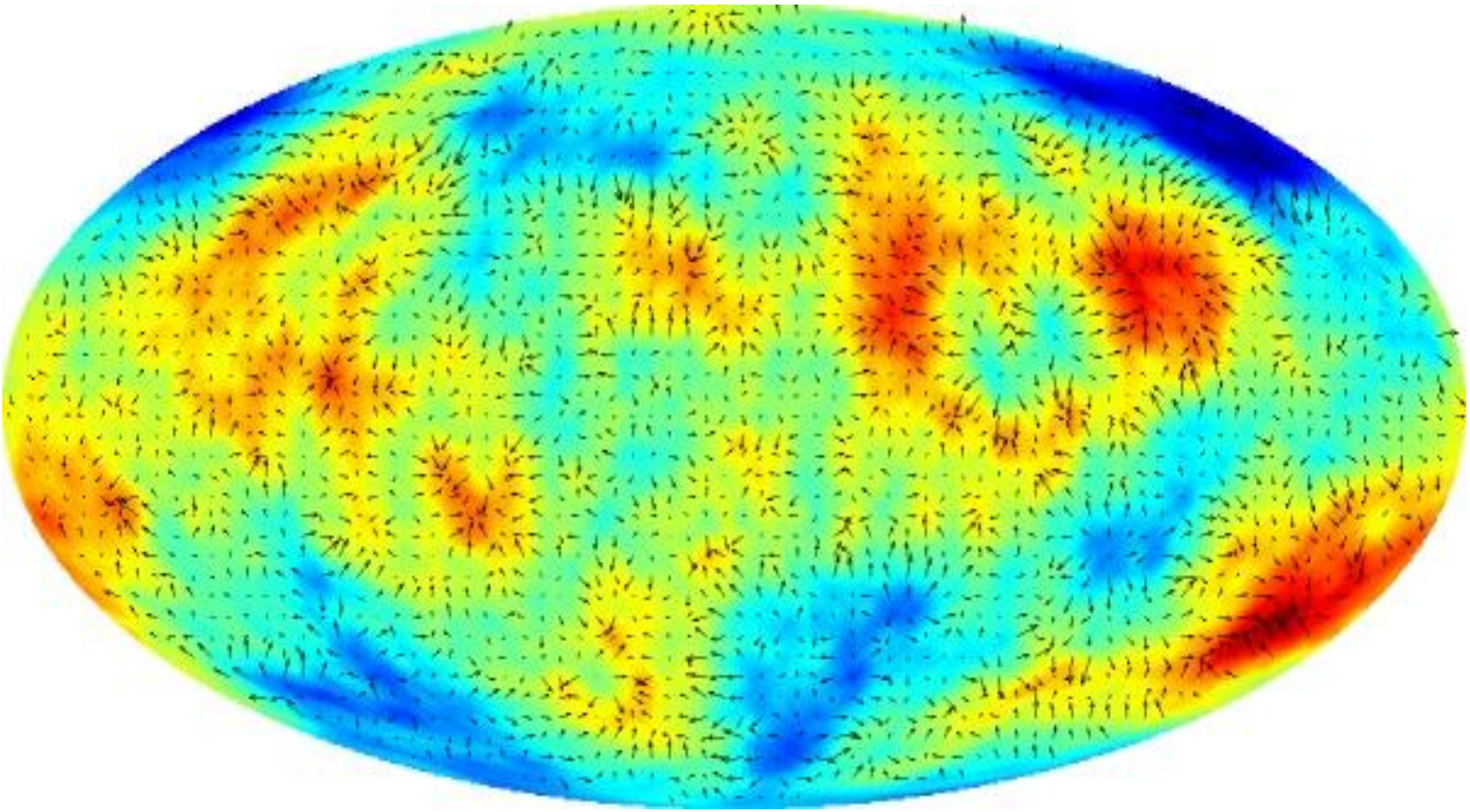
Can be computed with CLASS http://class-code.net or CAMB: http://camb.info

Redshift Dependence: broad redshift kernel all way along line of sight
Bulk of the signal $0.5 < z < 6$



Lensing potential and deflection angles: simulation

LensPix sky simulation code: http://cosmologist.info/lenspix (several others also available)



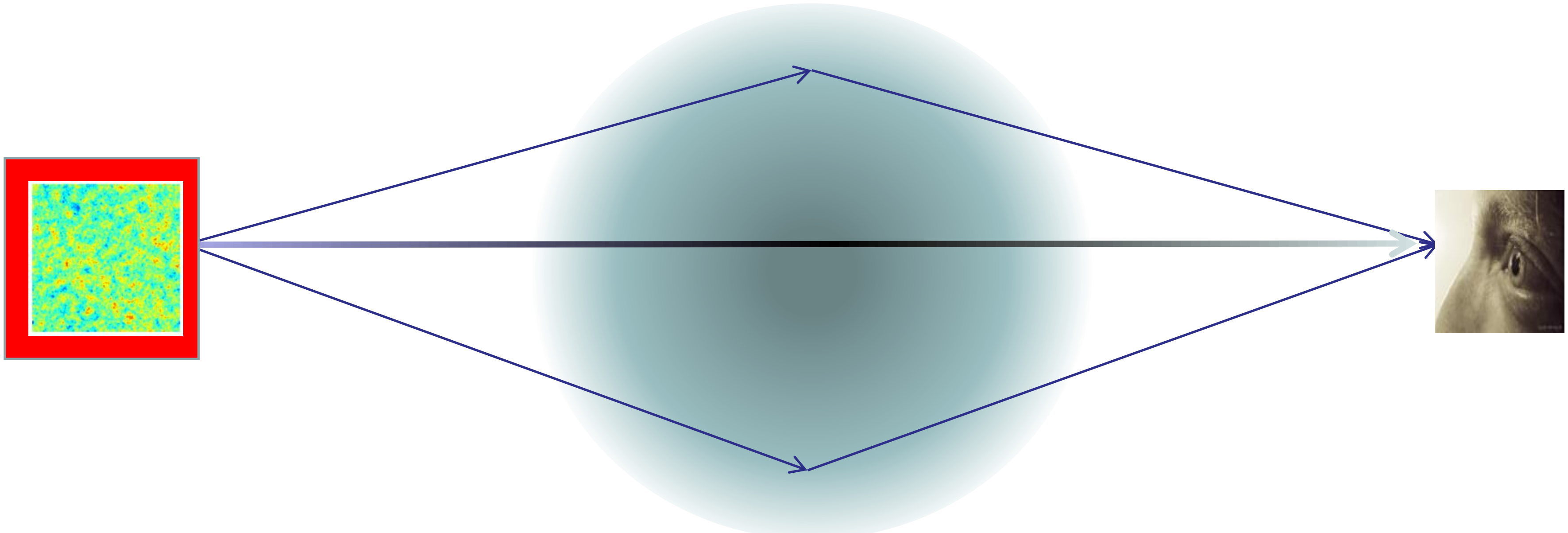
- Note: can only observe *lensed* sky

- Any bulk deflection is unobservable
  – degenerate with corresponding change in unlensed CMB:
  e.g.
  rotation of full sky
  translation in flat sky approximation

- Observations sensitive to *differences* of deflection angles
  - convergence and shear

Correlation with the CMB temperature

$$\Delta T_{\mathrm{ISW}}(\hat{\mathbf{n}}) = 2\int_0^{\chi_*} \mathrm{d}\chi\, \dot{\Psi}(\chi\hat{\mathbf{n}};\eta_0-\chi).$$

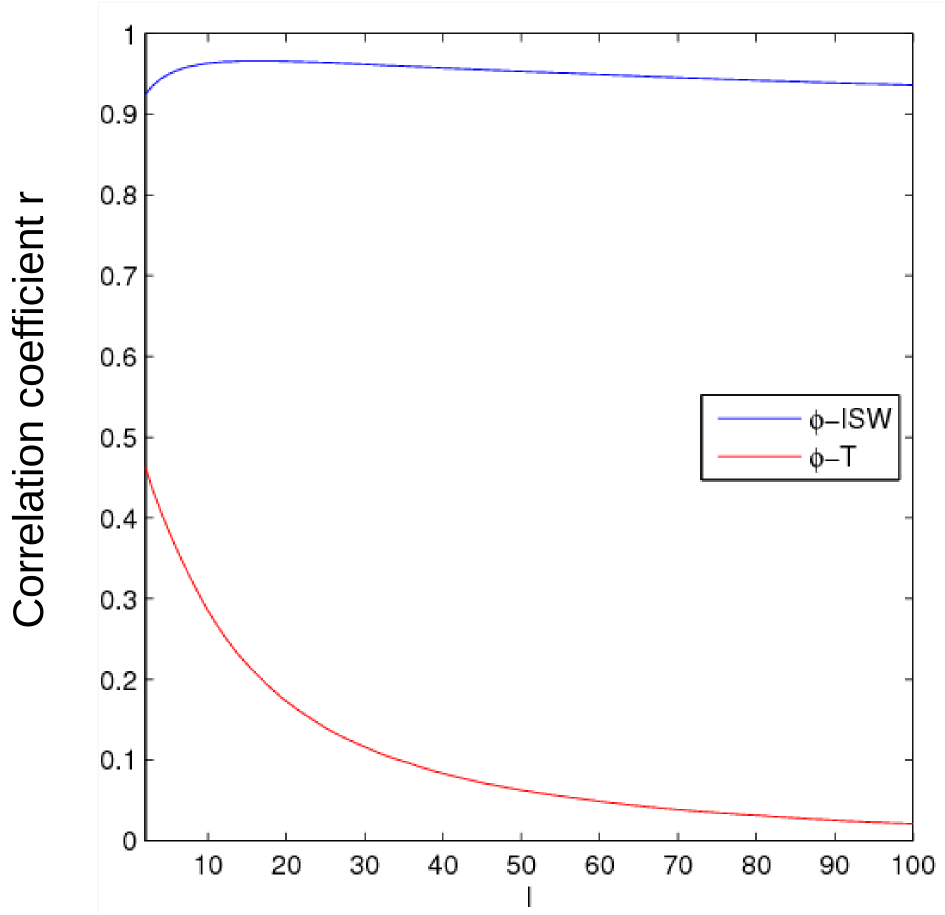


Overdensity: magnification correlated with positive Integrated Sachs-Wolfe (net blueshift)

Underdensity: demagnification correlated with negative Integrated Sachs-Wolfe (net redshift)

(small-scales: also SZ , Rees-Sciama..)

Lensing potential correlation with the CMB temperature



very small except on largest scales, mostly $l < 100$

Also small large-scale lensing–E polarization correlation (from reionization)

Calculating the lensed CMB power spectrum

- Approximations and assumptions:
  - Lensing potential uncorrelated to temperature
  - Gaussian lensing potential and temperature
  - Statistical isotropy

- Simplifying optional approximations
  - flat sky (good approximation)
  - series expansion (poor approximation, but still useful to understand)

Unlensed temperature field in flay sky approximation

- Fourier transforms:

$$\Theta(\mathbf{x}) = \int \frac{d^2\mathbf{l}}{2\pi}\Theta(\mathbf{l})e^{i\mathbf{l}\cdot\mathbf{x}}, \qquad \Theta(\mathbf{l}) = \int \frac{d^2\mathbf{x}}{2\pi}\Theta(\mathbf{x})e^{-i\mathbf{l}\cdot\mathbf{x}}.$$

- Statistical isotropy: $\langle\Theta(\mathbf{x})\Theta(\mathbf{x}')\rangle = \xi(|\mathbf{x}-\mathbf{x}'|).$

$$\begin{aligned}\langle\Theta(\mathbf{l})\Theta^*(\mathbf{l}')\rangle &= \int \frac{d^2\mathbf{x}}{2\pi}\int \frac{d^2\mathbf{x}'}{2\pi} e^{-i\mathbf{l}\cdot\mathbf{x}} e^{i\mathbf{l}'\cdot\mathbf{x}'}\xi(|\mathbf{x}-\mathbf{x}'|)\\ &= \int \frac{d^2\mathbf{x}}{2\pi}\int \frac{d^2\mathbf{r}}{2\pi} e^{i(\mathbf{l}'-\mathbf{l})\cdot\mathbf{x}} e^{i\mathbf{l}'\cdot\mathbf{r}}\xi(r)\\ &= \delta(\mathbf{l}'-\mathbf{l})\int d^2\mathbf{r}\, e^{i\mathbf{l}\cdot\mathbf{r}}\xi(r).\end{aligned}$$

So $\langle\Theta(\mathbf{l})\Theta^*(\mathbf{l}')\rangle = C_l^\Theta\delta(\mathbf{l}-\mathbf{l}').$ where $C_l^\Theta = \int d^2\mathbf{r}\, e^{i\mathbf{l}\cdot\mathbf{r}}\xi(r)$:

Similarly for the lensing potential (also assumed Gaussian and statistically isotropic)

Lensed field: series expansion approximation

$$\begin{aligned}\tilde{\Theta}(\mathbf{x}) &= \Theta(\mathbf{x}') = \Theta(\mathbf{x} + \boldsymbol{\nabla}\psi) \\ &\approx \Theta(\mathbf{x}) + \nabla^a\psi(\mathbf{x})\nabla_a\Theta(\mathbf{x}) + \tfrac{1}{2}\nabla^a\psi(\mathbf{x})\nabla^b\psi(\mathbf{x})\nabla_a\nabla_b\Theta(\mathbf{x}) + \ldots\end{aligned}$$

(BEWARE: this is not a very good approximation for power spectrum! see later)

Using Fourier transforms in flat sky approximation:

$$\boldsymbol{\nabla}\psi(\mathbf{x}) = i\int\frac{\mathrm{d}^2\mathbf{l}}{2\pi}\,\mathbf{l}\psi(\mathbf{l})e^{i\mathbf{l}\cdot\mathbf{x}}, \qquad \boldsymbol{\nabla}\Theta(\mathbf{x}) = i\int\frac{\mathrm{d}^2\mathbf{l}}{2\pi}\,\mathbf{l}\Theta(\mathbf{l})e^{i\mathbf{l}\cdot\mathbf{x}}$$

Then lensed harmonics then given by

use $\int d^2\mathbf{x}e^{i\mathbf{x}\cdot(\mathbf{l}_1-\mathbf{l}_2-\mathbf{l})} = (2\pi)^2\delta(\mathbf{l}_1-\mathbf{l}_2-\mathbf{l})$

$$\begin{aligned}\tilde{\Theta}(\mathbf{l}) \approx \Theta(\mathbf{l}) &- \int\frac{\mathrm{d}^2\mathbf{l}'}{2\pi}\,\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}')\Theta(\mathbf{l}') \\ &- \frac{1}{2}\int\frac{\mathrm{d}^2\mathbf{l}_1}{2\pi}\int\frac{\mathrm{d}^2\mathbf{l}_2}{2\pi}\,\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}]\,\mathbf{l}_1\cdot\mathbf{l}_2\Theta(\mathbf{l}_1)\psi(\mathbf{l}_2)\psi^*(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}).\end{aligned}$$

Lensed field still statistically isotropic: $\langle\tilde{\Theta}(\mathbf{l})\tilde{\Theta}^*(\mathbf{l}')\rangle = \delta(\mathbf{l}-\mathbf{l}')\tilde{C}_l^\Theta.$

with

$$\tilde{C}_l^\Theta \approx C_l^\Theta + \int \frac{d^2\mathbf{l}'}{(2\pi)^2}\left[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\right]^2 C^\psi_{|\mathbf{l}-\mathbf{l}'|}C^\Theta_{l'} - C_l^\Theta \int \frac{d^2\mathbf{l}'}{(2\pi)^2}(\mathbf{l}\cdot\mathbf{l}')^2 C^\psi_{l'}$$

Alternatively written as

$$\tilde{C}_l^\Theta \approx (1-l^2R^\psi)C_l^\Theta + \int \frac{d^2\mathbf{l}'}{(2\pi)^2}\left[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\right]^2 C^\psi_{|\mathbf{l}-\mathbf{l}'|}C^\Theta_{l'}.$$

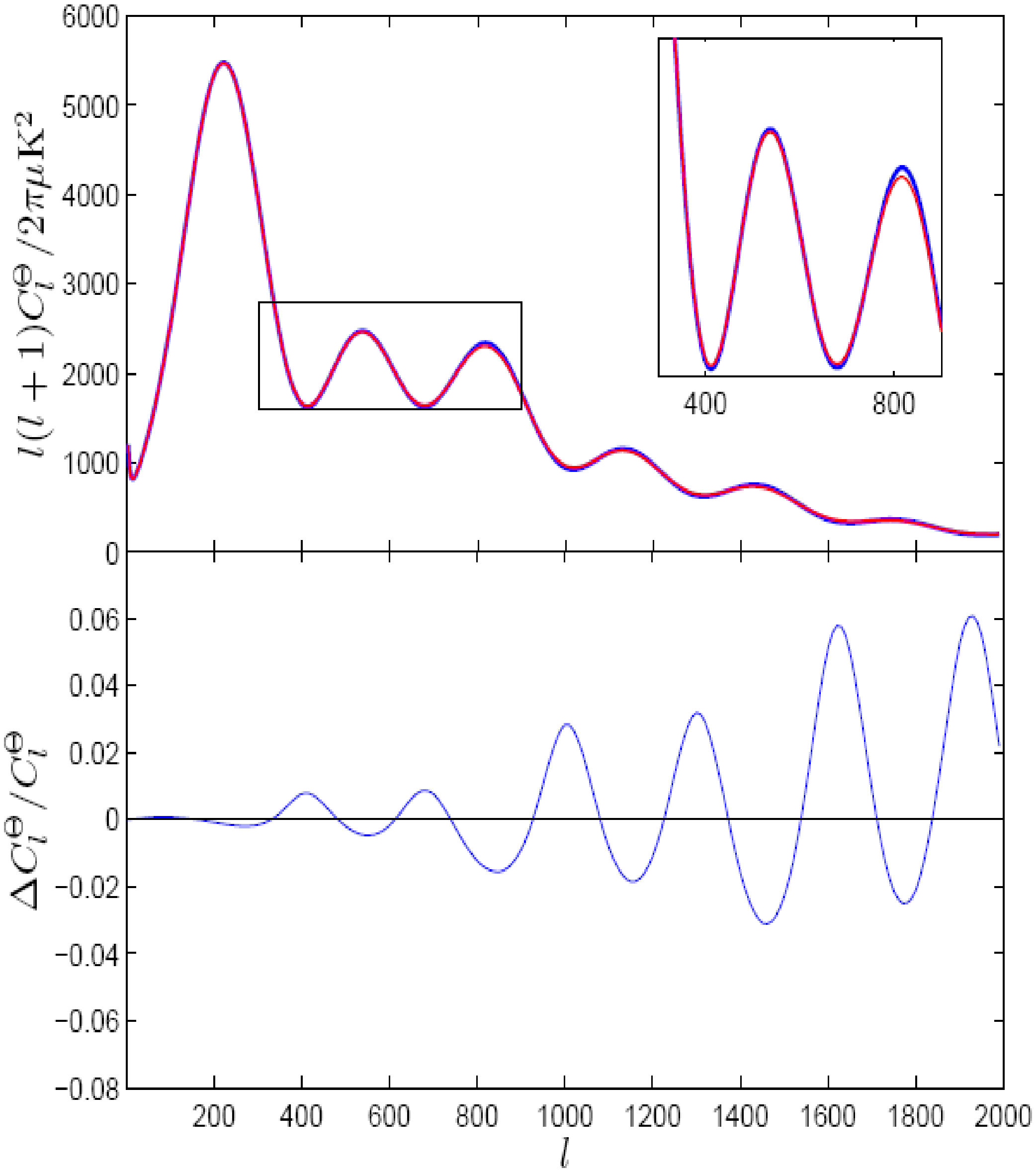
where $$R^\psi \equiv \frac{1}{2}\langle|\nabla\psi|^2\rangle = \frac{1}{4\pi}\int\frac{dl}{l}l^4C_l^\psi, \quad \sim 3\times 10^{-7}$$

(RMS deflection ~ 2.7 arcmin)

<u>Second term is a convolution with the deflection angle power spectrum</u>

- smoothes out acoustic peaks
- transfers power from large scales into the damping tail

Lensing effect on CMB temperature power spectrum



Small scales, large l limit:

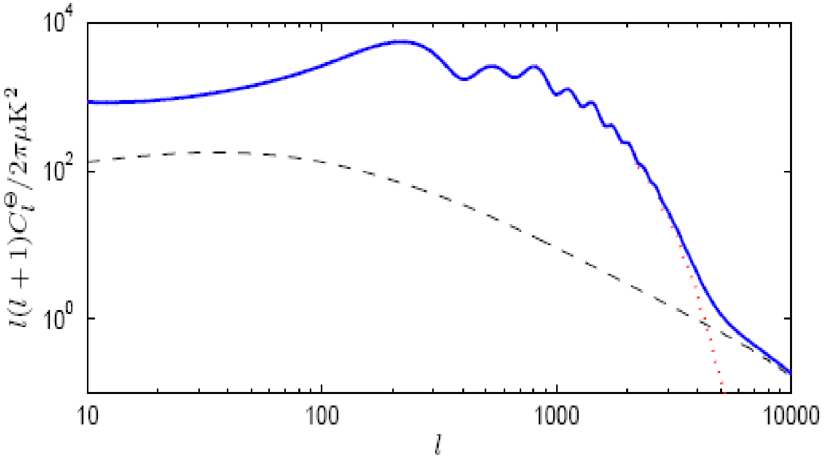
- unlensed CMB has very little power due to silk damping: $C_l^{\Theta} \sim 0$

$$\begin{aligned}\tilde{C}_l^{\Theta} &\approx \int \frac{\mathrm{d}^2\mathbf{l}'}{(2\pi)^2}\left[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\right]^2 C^{\psi}_{|\mathbf{l}'-\mathbf{l}|} C^{\Theta}_{l'} \\ &\approx C_l^{\psi}\int \frac{\mathrm{d}^2\mathbf{l}'}{(2\pi)^2}\left[\mathbf{l}'\cdot\mathbf{l}\right]^2 C^{\Theta}_{l'} \\ &\approx l^2 C_l^{\psi}\int \frac{\mathrm{d}l_2}{l_2}\frac{l_2^4 C^{\Theta}_{l_2}}{4\pi} \\ &\approx l^2 C_l^{\psi} R^{\Theta}.\end{aligned}$$

$$l' \ll l$$

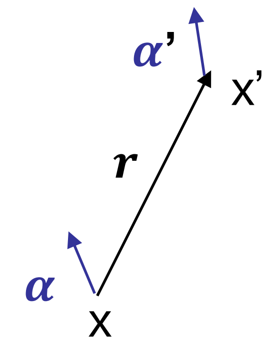
$$R^{\Theta} \equiv \frac{1}{2}\langle|\nabla T|^2\rangle = \frac{1}{4\pi}\int \frac{\mathrm{d}l}{l}\, l^4 C_l^{\Theta} \sim 10^9 \mu\mathrm{K}^2$$

- Proportional to the deflection angle power spectrum and the (scale independent) power in the gradient of the temperature



Better accurate calculation: lensed correlation function

- Do not perform series expansion



$$\tilde{\Theta}(\mathrm{x}) = \Theta(\mathrm{x} + \boldsymbol{\alpha})$$

Lensed correlation function:

$$\begin{aligned}
\tilde{\xi}(r) &\equiv \langle \tilde{\Theta}(\mathrm{x})\tilde{\Theta}(\mathrm{x}')\rangle \\
&= \langle \Theta(\mathrm{x}+\boldsymbol{\alpha})\Theta(\mathrm{x}'+\boldsymbol{\alpha}')\rangle \\
&= \int \frac{\mathrm{d}^2\mathbf{l}}{2\pi} \int \frac{\mathrm{d}^2\mathbf{l}'}{2\pi} \langle e^{i\mathbf{l}\cdot(\mathrm{x}+\boldsymbol{\alpha})} e^{-i\mathbf{l}'\cdot(\mathrm{x}'+\boldsymbol{\alpha}')}\rangle_{\boldsymbol{\alpha}} \langle \Theta(\mathbf{l})\Theta(\mathbf{l}')^*\rangle_{\Theta} \\
&= \int \frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2} C_l^{\Theta} e^{i\mathbf{l}\cdot\mathbf{r}} \langle e^{i\mathbf{l}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')}\rangle_{\boldsymbol{\alpha}}.
\end{aligned}$$

Assume uncorrelated

To calculate expectation value use

$$\begin{aligned}
\langle e^{iy}\rangle &= \frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty} \mathrm{d}y\, e^{iy} e^{-y^2/2\sigma_y^2} = \frac{1}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty} \mathrm{d}y\, e^{-(y-i\sigma_y^2)^2/2\sigma_y^2} e^{-\sigma_y^2/2} \\
&= e^{-\langle y^2\rangle/2}.
\end{aligned}$$

Seljak, astro-ph/9505109

$$\left\langle e^{i\mathbf{l}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')}\right\rangle = \exp\left(-\frac{1}{2}\left\langle[\mathbf{l}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')]^2\right\rangle\right)$$

where $$\begin{aligned}\left\langle[\mathbf{l}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')]^2\right\rangle &= l^i l^j \langle(\alpha_i-\alpha_i')(\alpha_j-\alpha_j')\rangle \\ &= l^2[C_{\rm gl}(0)-C_{\rm gl}(r)] + 2l^i l^j \hat{r}_{\langle i}\hat{r}_{j\rangle} C_{{\rm gl},2}(r) \\ &= l^2[\sigma^2(r)+\cos 2\phi\, C_{{\rm gl},2}(r)].\end{aligned}$$

Have defined:

$$\begin{aligned}C_{\rm gl}(r) &\equiv \langle \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}'\rangle \\ &= \int \frac{\mathrm{d}^2\mathbf{l}}{2\pi}\, l^2 C_l^\psi e^{i\mathbf{l}\cdot\mathbf{r}} \\ &= \frac{1}{2\pi}\int \mathrm{d}l\, l^3 C_l^\psi J_0(lr),\end{aligned}$$

$$\sigma^2(r) \equiv C_{\rm gl}(0) - C_{\rm gl}(r) = \tfrac{1}{2}\langle(\boldsymbol{\alpha}-\boldsymbol{\alpha}')^2\rangle$$

- variance of the *difference* of deflection angles

$$\begin{aligned}C_{{\rm gl},2}(r) &\equiv -2\hat{r}_{\langle i}\hat{r}_{j\rangle}\langle \alpha^i \alpha'^j\rangle \\ &= -2\int \frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\hat{r}_{\langle i}\hat{r}_{j\rangle} l^i l^j C_l^\psi e^{i\mathbf{l}\cdot\mathbf{r}} \\ &= -2\int \frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\frac{l^2\cos 2\phi}{2} C_l^\psi e^{i\mathbf{l}\cdot\mathbf{r}} \\ &= \frac{1}{2\pi}\int \mathrm{d}l\, l^3 C_l^\psi J_2(lr).\end{aligned}$$

small correction from transverse differences

So lensed correlation function is

$$\tilde{\xi}(r) = \int \frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2} C_l^\Theta e^{i\mathbf{l}\cdot\mathbf{r}} \exp\left(-\frac{1}{2}\langle[\mathbf{l}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}')]^2\rangle\right)$$
$$= \int \frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2} C_l^\Theta e^{ilr\cos\phi} \exp\left(-\frac{1}{2}l^2[\sigma^2(r) + \cos 2\phi\, C_{\mathrm{gl},2}(r)]\right)$$

Expand exponential using

$$e^{-r\cos\phi} = \sum_{n=-\infty}^{\infty}(-1)^n I_n(r) e^{in\phi} = I_0(r) + 2\sum_{n=1}^{\infty}(-1)^n I_n(r)\cos(n\phi)$$

Integrate over angles gives final result:

$$\tilde{\xi}(r) = \int \frac{\mathrm{d}l}{l}\frac{l^2 C_l^\Theta}{2\pi} e^{-l^2\sigma^2(r)/2} \sum_{n=-\infty}^{\infty} I_n\left[l^2 C_{\mathrm{gl},2}(r)/2\right] J_{2n}(lr)$$
$$= \int \frac{\mathrm{d}l}{l}\frac{l^2 C_l^\Theta}{2\pi} e^{-l^2\sigma^2(r)/2}\left[J_0(lr) + \frac{1}{2}l^2 C_{\mathrm{gl},2}(r) J_2(lr) + \dots\right]$$

Note exponential: non-perturbative in lensing potential

Power spectrum and correlation function related by

$$C_l^\Theta = \int \mathrm{d}^2\mathbf{r} e^{i\mathbf{l}\cdot\mathbf{r}}\xi(r) = \int r\mathrm{d}r \int \mathrm{d}\phi_\mathbf{r}\, e^{ilr\cos(\phi_\mathbf{l}-\phi_\mathbf{r})}\xi(r) = 2\pi \int r\mathrm{d}r\, J_0(lr)\xi(r)$$

used Bessel functions defined by

$$e^{ir\cos\phi} = \sum_{n=-\infty}^{\infty} i^n J_n(r) e^{in\phi} = J_0(r) + 2\sum_{n=1}^{\infty} i^n J_n(r)\cos(n\phi).$$

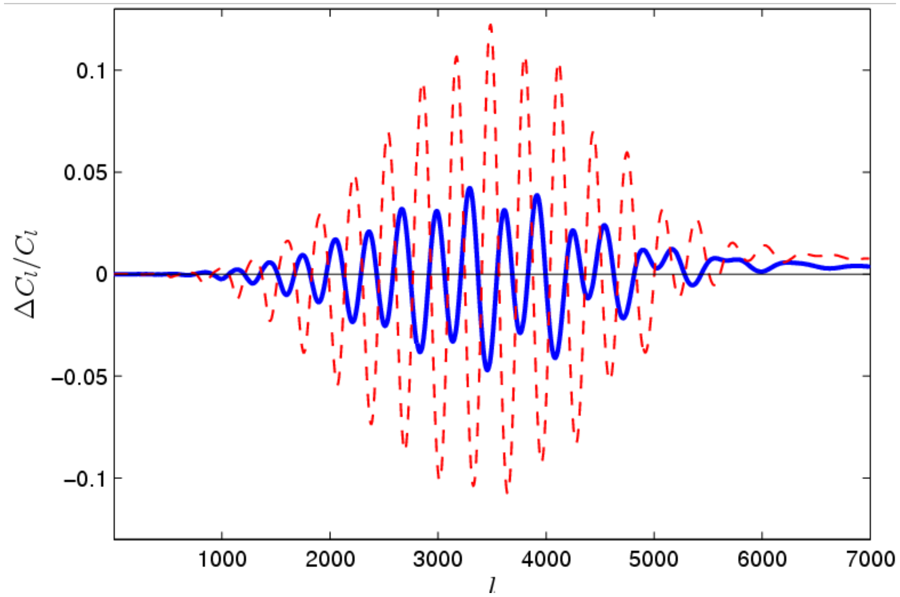
Can be generalized to fully spherical calculation: see review, astro-ph/0601594
However flat sky accurate to <~ 1% on the lensed power spectrum

Series expansion in deflection angle?

$$\tilde{\Theta}(\mathbf{x}) = \Theta(\mathbf{x}') = \Theta(\mathbf{x} + \boldsymbol{\nabla}\psi)$$
$$\approx \Theta(\mathbf{x}) + \nabla^a\psi(\mathbf{x})\nabla_a\Theta(\mathbf{x}) + \frac{1}{2}\nabla^a\psi(\mathbf{x})\nabla^b\psi(\mathbf{x})\nabla_a\nabla_b\Theta(\mathbf{x}) + \ldots$$

Only a good approximation when:
- deflection angle much smaller than wavelength of temperature perturbation
- OR, very small scales where temperature is close to a gradient

CMB lensing is a very specific physical second order effect; not accurately contained in 2nd order expansion – differs by significant 3rd and higher order terms



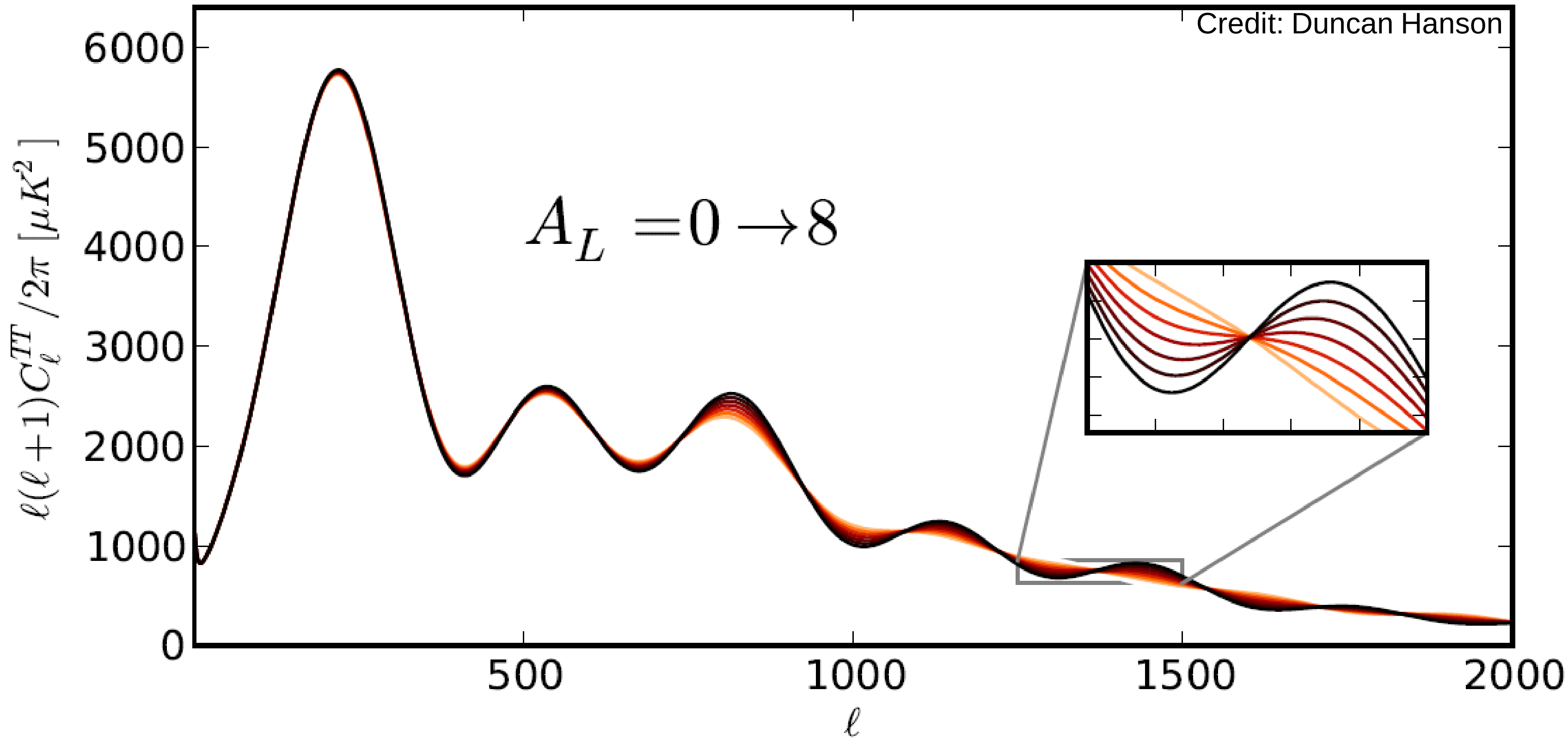
Error using series expansion:

temperature
E-polarization

Series expansion only good on large and very small scales – don't use for lensed C_l

Consistency check, is amount of smoothing at the expected level: $A_L = 1$?

A_L defined so that lensing smoothing calculated using $A_L C_l^{\psi\psi}$ rather than physical $C_l^{\psi\psi}$



- Can we detect preference for $A_L > 0$ in the data?
- Marginalizing over A_L is also a way of “removing” lensing information from C_l

LCDM parameter constraints from Planck alone

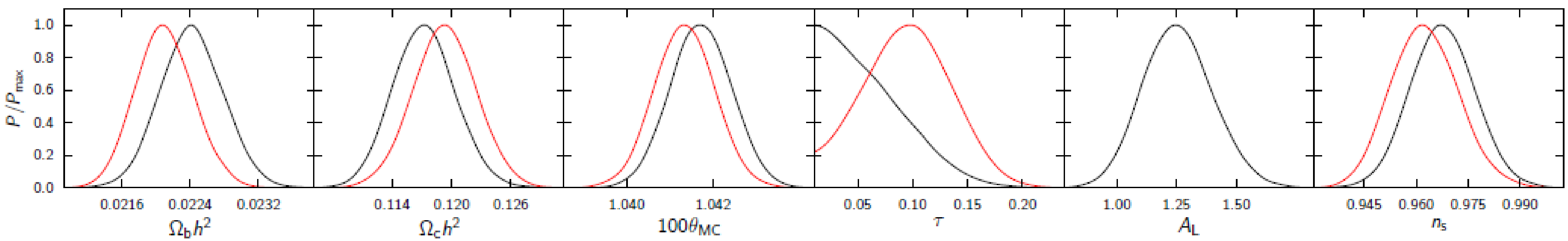
(no WMAP polarization)

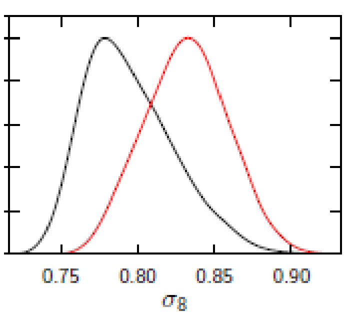
Lensing in the power spectrum
is detected at high significance:

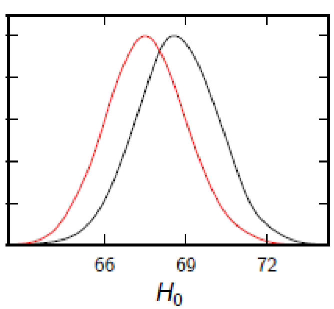
$$A_L = 1.28^{+0.29}_{-0.26} \; (2\sigma)$$

(actually a bit high??)

$A_L = 1$ (physical, main result) A_L varying (non-physical)







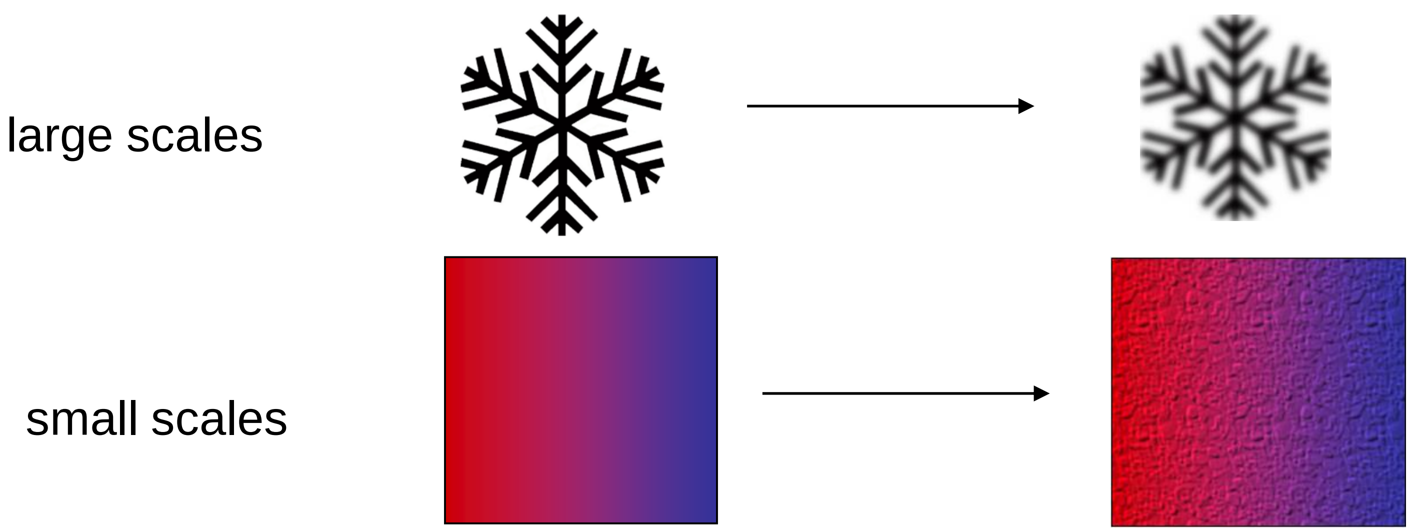
Marginalizing A_L (~ remove lensing information)
significantly shifts parameters
With lensing – almost a constraint on $\tau > 0$

Oddity: power spectrum data seems to like high A_L

Summary so far

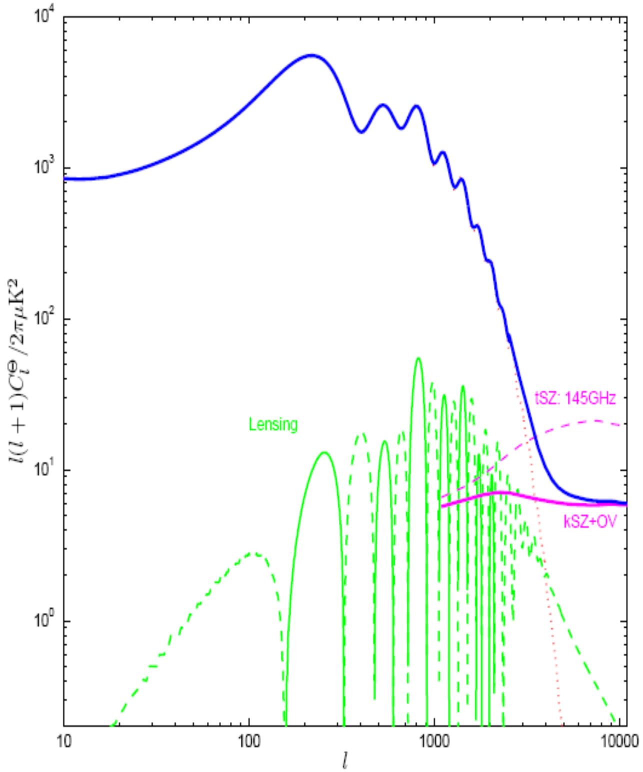
- Deflection angles of ~ 3 arcminutes, but correlated on degree scales

- Lensing convolves TT with deflection angle power spectrum
  - Acoustic peaks slightly blurred
  - Power transferred to small scales



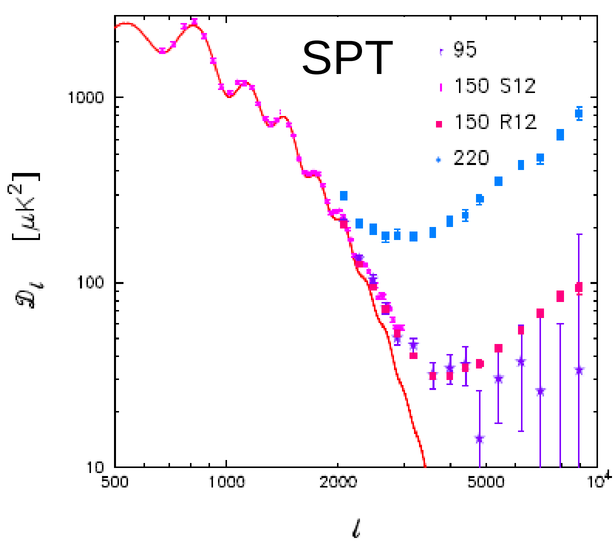
Other specific non-linear effects

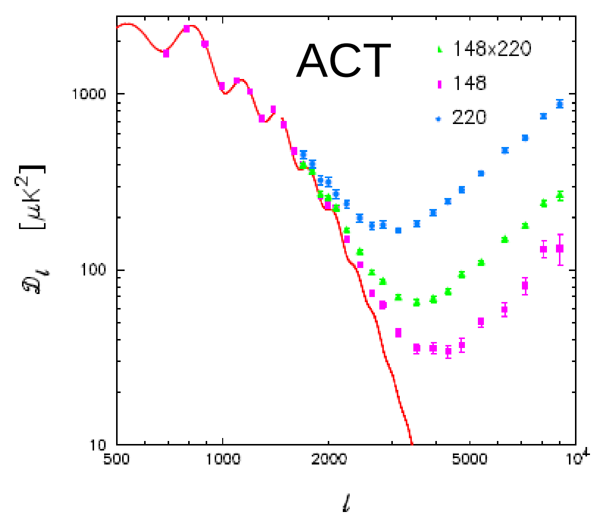
- Thermal Sunyaev-Zeldovich
  Inverse Compton scattering from hot gas: frequency dependent signal
- Kinetic Sunyaev-Zeldovich (kSZ)
  Doppler from bulk motion of clusters; patchy reionization; (almost) frequency independent signal
- Ostriker-Vishniac (OV)
  same as kSZ but for early linear bulk motion
- Rees-Sciama
  Integrated Sachs-Wolfe from evolving non-linear potentials: frequency independent



Lensing important at $500 < l < 3000$
Dominated by SZ, CIB etc. on small scales

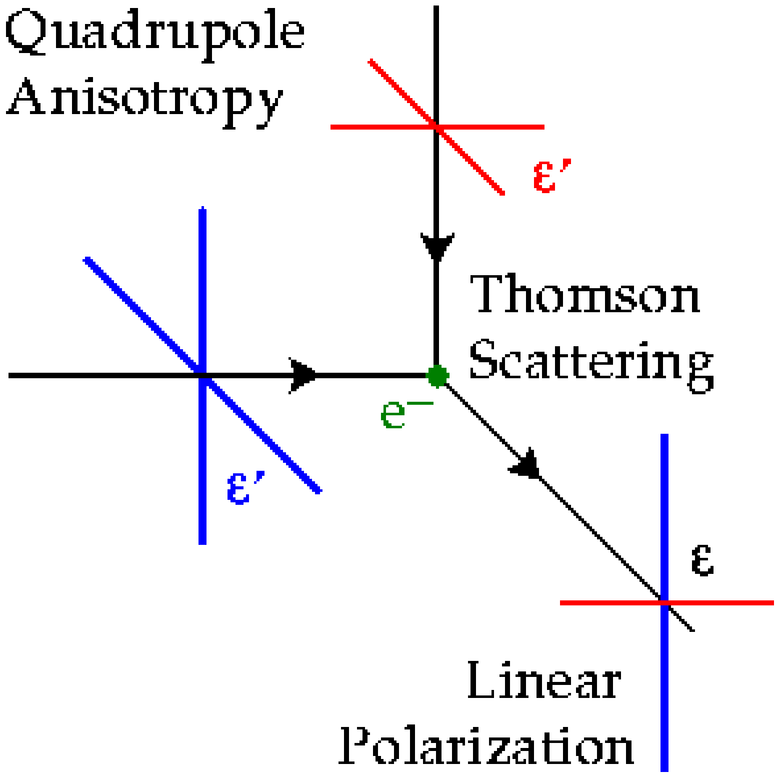
+ foregrounds
- actually dominate at $l \gg 2000$





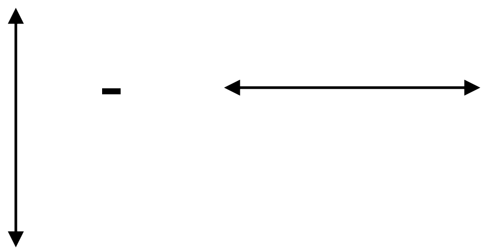
CMB Polarization

Generated during last scattering (and reionization) by Thomson scattering of anisotropic photon distribution

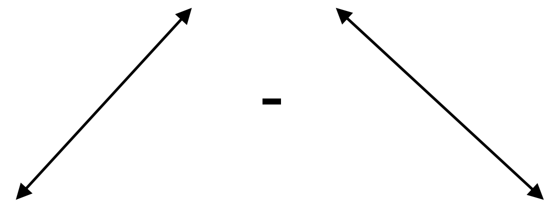


Hu astro-ph/9706147

Observed Stokes' Parameters



Q



U

Q → -Q, U → -U under 90 degree rotation

Q → U, U → -Q under 45 degree rotation

Measure E field perpendicular to observation direction $\hat{\boldsymbol{n}}$
Intensity matrix defined as

$$\mathcal{P}_{ab} = C\langle E_a E_b^* \rangle = P_{ab} + \frac{1}{2}\delta_{ab} I + V_{[ab]}$$

Linear polarization + Intensity + circular polarization

CMB only linearly polarized. In some fixed basis

$$P_{ij} = \frac{1}{2}\begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}$$

Alternative complex representation

Define complex vectors $\mathrm{e}_{\pm} = \mathrm{e}_1 \pm i\mathrm{e}_2$ e.g. $\mathrm{e}_{\pm} = \mathrm{e}_x \pm i\mathrm{e}_y$

And complex polarization

$$P \equiv \mathrm{e}_+^a \mathrm{e}_+^b P_{ab} = Q + iU$$
$$P^* = \mathrm{e}_-^a \mathrm{e}_-^b P_{ab} = Q - iU.$$

Under a rotation of the basis vectors

$$\begin{aligned}\mathrm{e}_{\pm} \equiv \mathrm{e}_x \pm i\mathrm{e}_y &\to \mathrm{e}_x' \pm i\mathrm{e}_y' \\ &= (\cos\gamma\,\mathrm{e}_x - \sin\gamma\,\mathrm{e}_y) \pm i(\sin\gamma\,\mathrm{e}_x + \cos\gamma\,\mathrm{e}_y) \\ &= e^{\pm i\gamma}(\mathrm{e}_x \pm i\mathrm{e}_y) = e^{\pm i\gamma}\mathrm{e}_{\pm}.\end{aligned}$$

$$P' = \mathrm{e}_+^{a\,\prime} \mathrm{e}_+^{b\,\prime} P_{ab} = e^{2i\gamma} P.$$ - spin 2 field

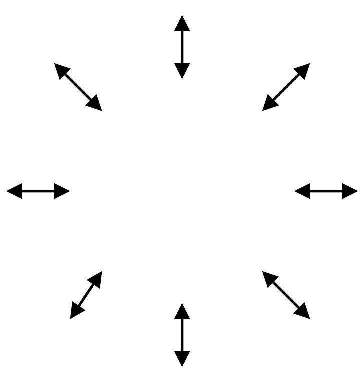
E and B polarization

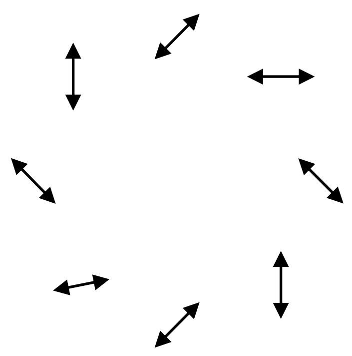
$$\mathcal{P}_{ab} = \nabla_{\langle a}\nabla_{b\rangle} P_E - \epsilon^{c}{}_{\langle a}\nabla_{b\rangle}\nabla_c P_B$$

“gradient” modes
E polarization

“curl” modes
B polarization

e.g.





E and B harmonics

- Expand scalar P_E and P_B in scalar harmonics $Q_k = e^{i\mathbf{l}\cdot\mathbf{x}}$
- Expand P in spin-2 harmonics

$$P: = \sum_k (E_k + iB_k)_2 Q_k \qquad {}_2Q_k \equiv \frac{N_k}{\sqrt{2}} \mathrm{e}_+^a \mathrm{e}_+^b \nabla_a \nabla_b Q_k$$

$$P^* = \sum_k (E_k - iB_k)_{-2} Q_k \qquad {}_{-2}Q_k \equiv \frac{N_k}{\sqrt{2}} \mathrm{e}_-^a \mathrm{e}_-^b \nabla_a \nabla_b Q_k$$

Harmonics are orthogonal over the full sky:

E/B decomposition is exact and lossless on the full sky

Zaldarriaga, Seljak: astro-ph/9609170
Kamionkowski, Kosowsky, Stebbins: astro-ph/9611125

On the flat sky spin-2 harmonics are

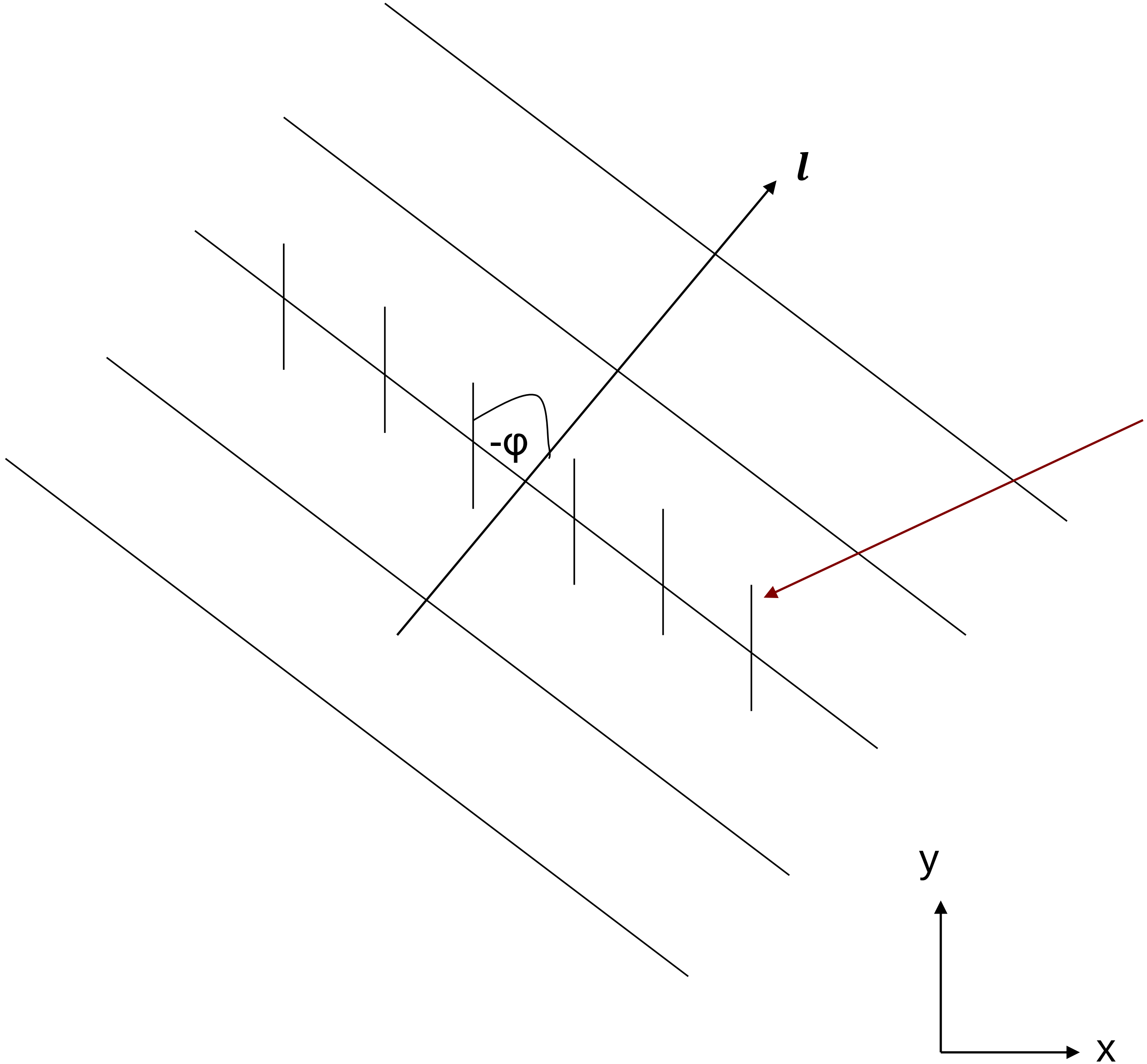
$$
\begin{aligned}
{}_2Q(\mathbf{l}) &= l^{-2}\, \mathbf{e}_+^a \mathbf{e}_+^b \nabla_a \nabla_b e^{i\mathbf{l}\cdot\mathbf{x}} \\
&= l^{-2}\, (\partial_x + i\partial_y)^2 e^{i\mathbf{l}\cdot\mathbf{x}} \\
&= -(\cos\phi_{\mathbf{l}} + i\sin\phi_{\mathbf{l}})^2 e^{i\mathbf{l}\cdot\mathbf{x}} \\
&= -e^{2i\phi_{\mathbf{l}}} e^{i\mathbf{l}\cdot\mathbf{x}}.
\end{aligned}
$$

$$
\begin{aligned}
P = Q + iU &= -\int \frac{d^2\mathbf{l}}{2\pi} \left[(E(\mathbf{l}) + iB(\mathbf{l})\right] e^{2i\phi_{\mathbf{l}}} e^{i\mathbf{l}\cdot\mathbf{x}} \\
P^* = Q - iU &= -\int \frac{d^2\mathbf{l}}{2\pi} \left[(E(\mathbf{l}) - iB(\mathbf{l})\right] e^{-2i\phi_{\mathbf{l}}} e^{i\mathbf{l}\cdot\mathbf{x}}
\end{aligned}
$$

Inverse relations:

$$
\begin{aligned}
E(\mathbf{l}) + iB(\mathbf{l}) &= -\int \frac{d^2\mathbf{x}}{2\pi} P e^{-2i\phi_{\mathbf{l}}} e^{-i\mathbf{l}\cdot\mathbf{x}} \\
E(\mathbf{l}) - iB(\mathbf{l}) &= -\int \frac{d^2\mathbf{x}}{2\pi} P^* e^{2i\phi_{\mathbf{l}}} e^{-i\mathbf{l}\cdot\mathbf{x}}.
\end{aligned}
$$

Factors of $e^{\pm 2i\phi_{\mathbf{l}}}$ rotate polarization to physical frame defined by wavenumber **l**



Polarization
$Q_{xy}=-1$, $U_{xy}=0$
$P_{xy} = -1$

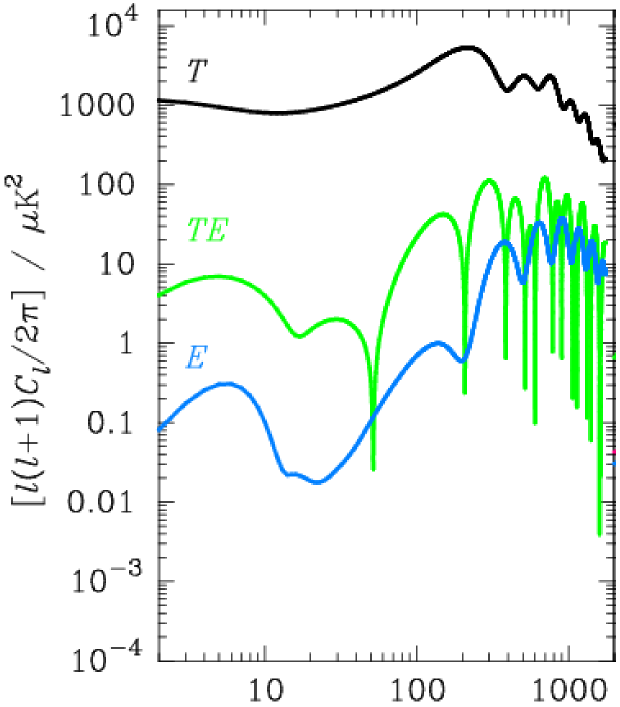
in bases wrt
(rotated by $-\varphi$)
$Q_{\boldsymbol{l}} = 0$, $U_{\boldsymbol{l}} = 1$
$P_{\boldsymbol{l}} = i$

$P_{\boldsymbol{l}} = P_{xy}\ e^{-2i\varphi}$

CMB Polarization Signals

- E polarization from scalar, vector and tensor modes
- B polarization only from vector and tensor modes (curl grad = 0) + non-linear scalars

Average over possible realizations (statistically isotropic):



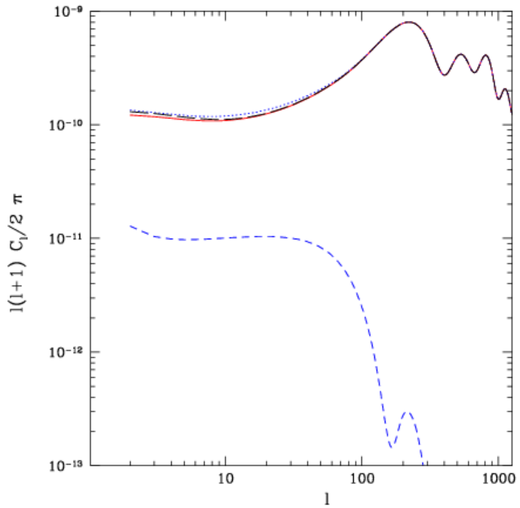
$$\langle E(\mathbf{l})E^*(\mathbf{l}')\rangle = \delta(\mathbf{l}-\mathbf{l}')C_l^E \qquad \langle B(\mathbf{l})B^*(\mathbf{l}')\rangle = \delta(\mathbf{l}-\mathbf{l}')C_l^B$$

$$\langle E(\mathbf{l})\Theta^*(\mathbf{l}')\rangle = \delta(\mathbf{l}-\mathbf{l}')C_l^X$$

Expected signal from scalar modes

Primordial Gravitational Waves
(tensor modes)

- Well motivated by some inflationary models
  - Amplitude measures inflaton potential at horizon crossing
  - distinguish models of inflation
- Observation would rule out other models
- Weakly constrained from CMB temperature anisotropy



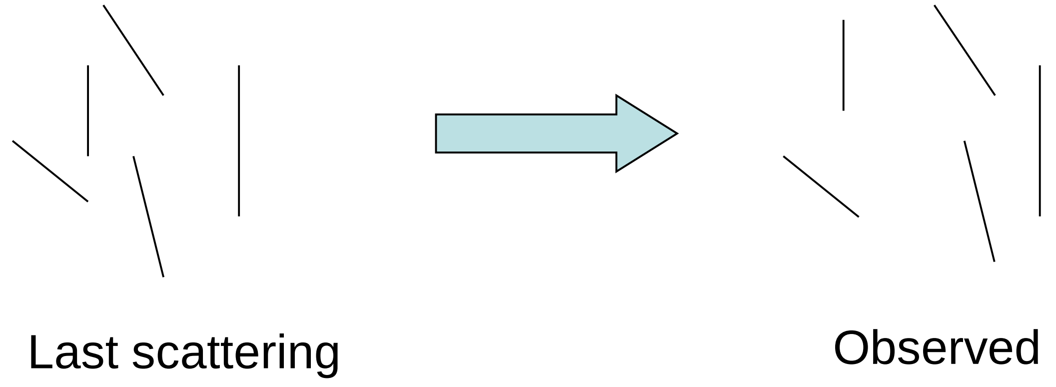
- Anisotropic redshifting of 0th order last scattering by 1st order gravitational waves along the line of sight
- cosmic variance limited to 10%
- degenerate with other parameters (tilt, reionization, etc)

Look at CMB polarization:
'B-mode' smoking gun

Lensing of polarization

- Polarization not rotated w.r.t. parallel transport (vacuum is not birefringent)
- Q and U Stokes parameters simply re-mapped by the lensing deflection field

e.g.



Series expansion

Similar to temperature derivation, but now complex spin-2 quantities:

$$\tilde{P}(\mathbf{x}) = P(\mathbf{x}+\nabla\psi) \sim P(\mathbf{x}) + \nabla^a\psi\nabla_b P(\mathbf{x}) + \frac{1}{2}\nabla^c\psi\nabla^d\psi\nabla_c\nabla_d P(\mathbf{x})$$

Unlensed B is expected to be very small. Simplify by setting to zero.
Expand in harmonics

$$\tilde{E}(\mathbf{l}) \pm i\tilde{B}(\mathbf{l}) \approx E(\mathbf{l}) - \int \frac{\mathrm{d}^2\mathbf{l}'}{2\pi}\,\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\psi(\mathbf{l}-\mathbf{l}')E(\mathbf{l}')$$
$$-\frac{1}{2}\int\frac{\mathrm{d}^2\mathbf{l}_1}{2\pi}\int\frac{\mathrm{d}^2\mathbf{l}_2}{2\pi}\,e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}]\,\mathbf{l}_1\cdot\mathbf{l}_2 E(\mathbf{l}_1)\psi(\mathbf{l}_2)\psi^*(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})$$

First order terms are

$$\tilde{E}(l) = E(l) - \int \frac{d^2 l'}{2\pi} l' \cdot (l-l') \cos(2[\phi_{l'}-\phi_l])\,\psi(l-l')E(l')$$

$$\tilde{B}(l) = -\int \frac{d^2 l'}{2\pi} l' \cdot (l-l') \sin(2[\phi_{l'}-\phi_l])\,\psi(l-l')E(l')$$

Lensed spectrum: lowest order calculation

Need second order expansion for consistency with lensed E 0^{th} x 2^{nd} order + 1^{st} x 1^{st} order :

$$\tilde{E}(\mathbf{l}) \pm i\tilde{B}(\mathbf{l}) \approx E(\mathbf{l}) - \int \frac{\mathrm{d}^2\mathbf{l}'}{2\pi}\, \mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}') e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})} \psi(\mathbf{l}-\mathbf{l}')E(\mathbf{l}')$$
$$-\frac{1}{2}\int \frac{\mathrm{d}^2\mathbf{l}_1}{2\pi}\int \frac{\mathrm{d}^2\mathbf{l}_2}{2\pi} e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})} \mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}]\, \mathbf{l}_1\cdot\mathbf{l}_2 E(\mathbf{l}_1)\psi(\mathbf{l}_2)\psi^*(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})$$

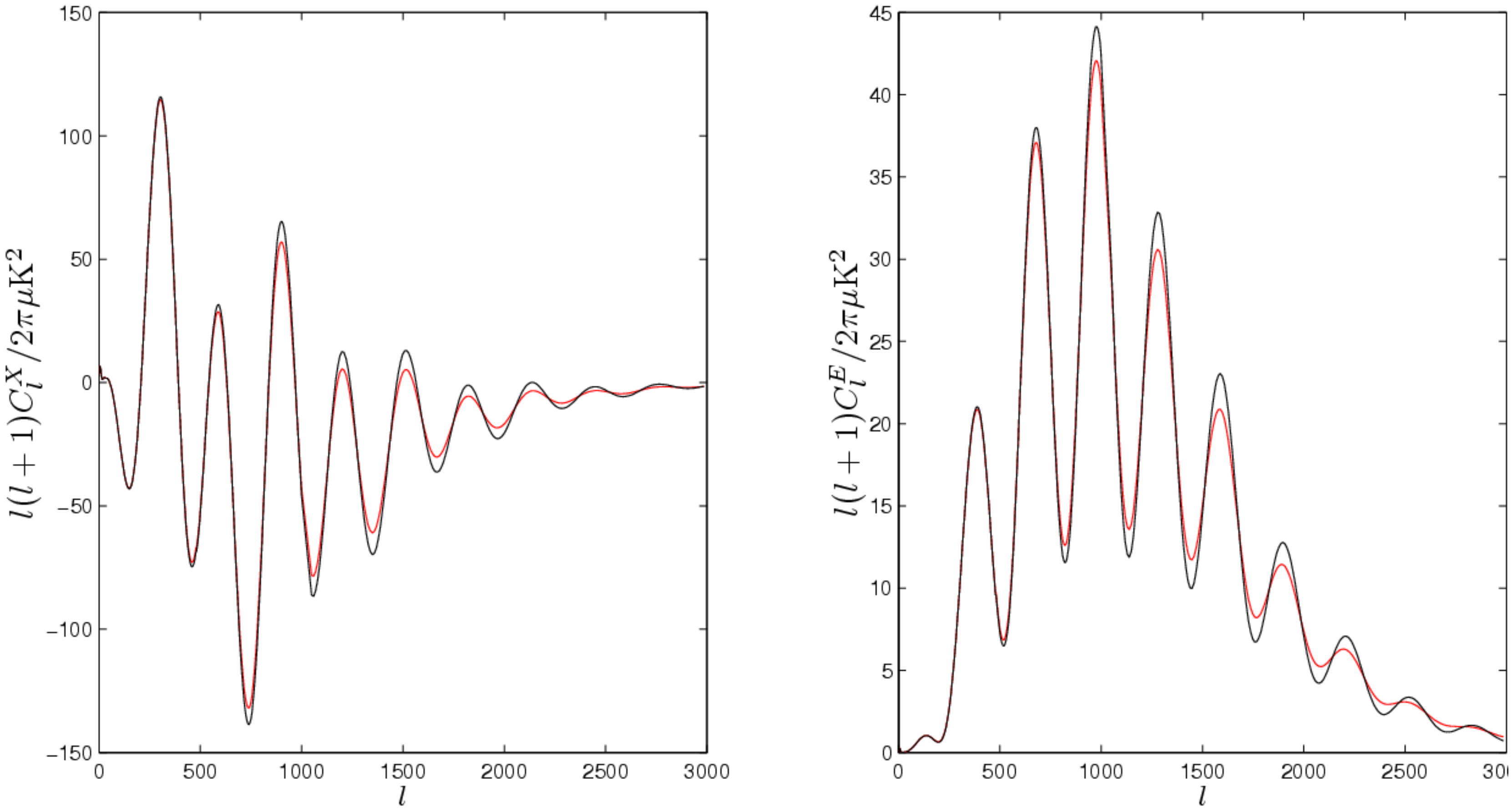
Calculate power spectrum. Result is

$$\tilde{C}_l^E = (1 - l^2 R^\psi)C_l^E + \int \frac{\mathrm{d}^2\mathbf{l}'}{(2\pi)^2} \left[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\right]^2 C^\psi_{|\mathbf{l}-\mathbf{l}'|} C^E_{|\mathbf{l}'|} \cos^2 2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})$$

$$\tilde{C}_l^B = \int \frac{\mathrm{d}^2\mathbf{l}'}{(2\pi)^2} \left[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\right]^2 C^\psi_{|\mathbf{l}-\mathbf{l}'|} C^E_{|\mathbf{l}'|} \sin^2 2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})$$

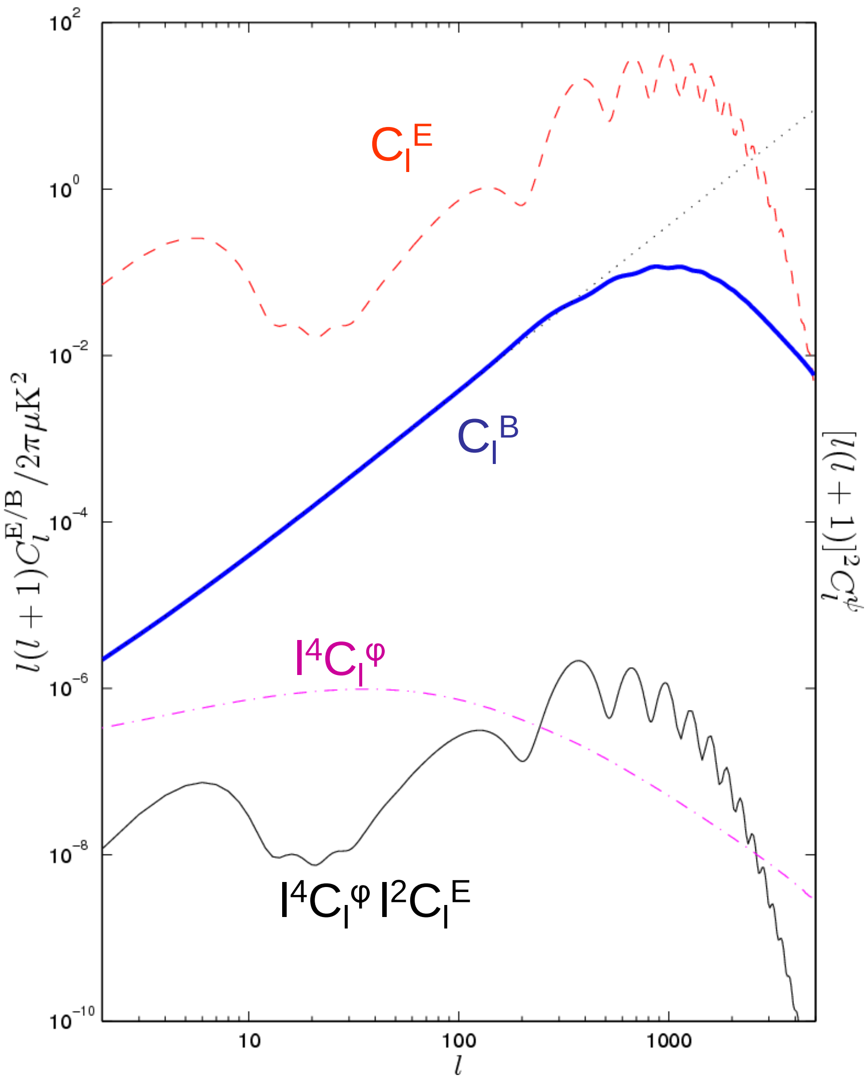
$$\tilde{C}_l^X = (1 - l^2 R^\psi)C_l^X + \int \frac{\mathrm{d}^2\mathbf{l}'}{(2\pi)^2} \left[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\right]^2 C^\psi_{|\mathbf{l}-\mathbf{l}'|} C^X_{|\mathbf{l}'|} \cos 2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}}).$$

Effect on EE and TE similar to temperature: convolution smoothing + transfer of power to small scales



Polarization lensing power spectra

BB generated by lensing even if unlensed B=0



On large scales, $|\mathbf{l}| \ll |\mathbf{l}'|$ lensed BB given by

$$\tilde{C}_l^B \sim \int \frac{d^2\mathbf{l}'}{(2\pi)^2} \, l'^4 C_{l'}^\psi \, C_{l'}^E \sin^2 2(\phi_{\mathbf{l}'} - \phi_{\mathbf{l}})$$

$$= \frac{1}{4\pi} \int \frac{dl'}{l'} \, l'^4 C_{l'}^\psi \; l'^2 C_{l'}^E,$$

Nearly white spectrum on large scales
(power spectrum independent of l)

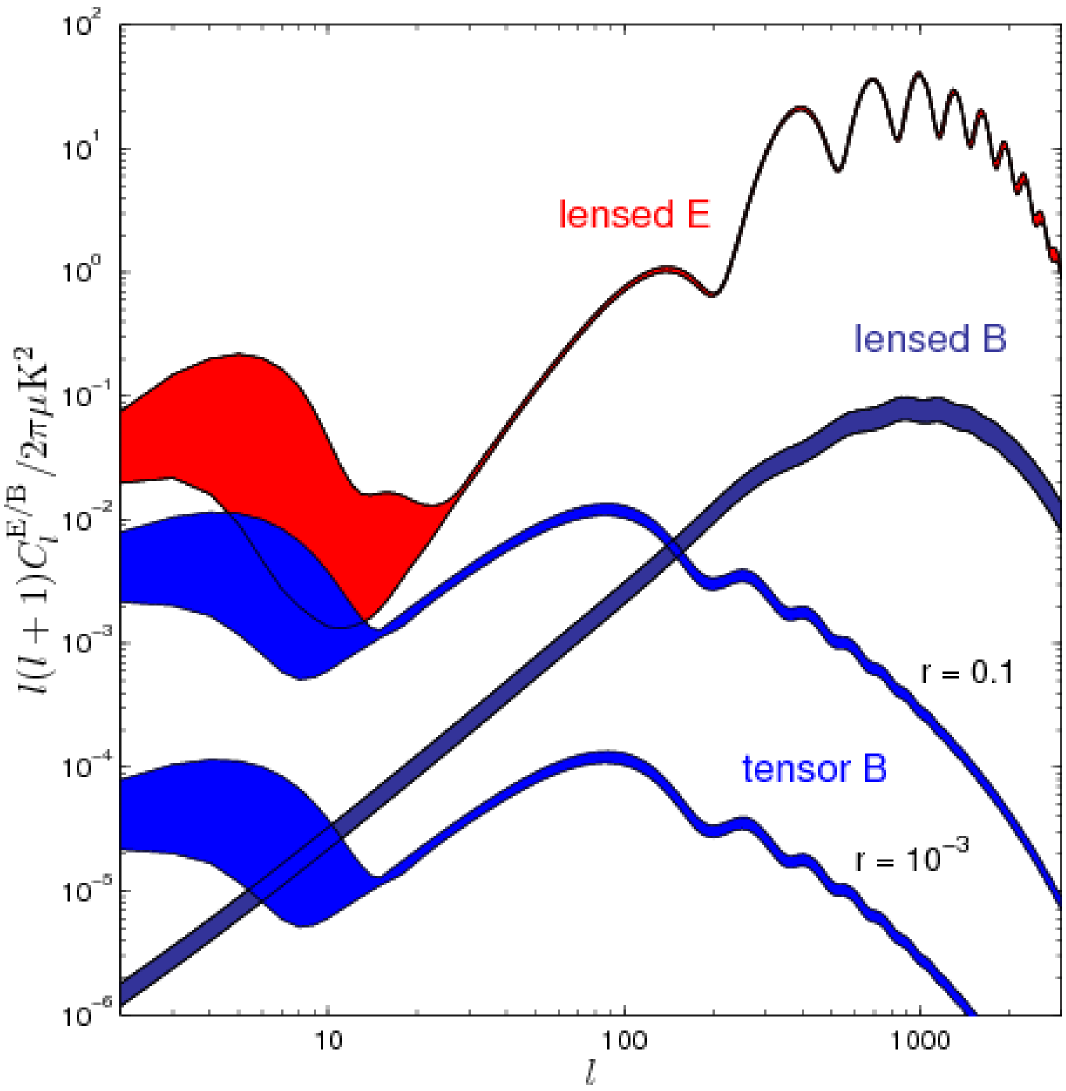
$$\tilde{C}_l^B \sim 2 \times 10^{-6} \mu\mathrm{K}^2$$

- unless removed, acts like an effective white-noise of $5\ \mu\mathrm{K}$ arcmin

Can also do more accurate calculation using polarization correlation functions

Polarization power spectra

Current 95% indirect limits for LCDM given WMAP+2dF+HST



Note: foregrounds expected to be much smaller than T for small-scale polarization: good!

arXiv: 1307.5830

Detection of B-mode Polarization in the Cosmic Microwave Background with Data from the South Pole Telescope

D. Hanson,[1] S. Hoover,[2,3] A. Crites,[2,4] P. A. R. Ade,[5] K. A. Aird,[6] J. E. Austermann,[7] J. A. Beall,[8] A. N. Bender,[1] B. A. Benson,[2,3] L. E. Bleem,[2,9] J. J. Bock,[10,11] J. E. Carlstrom,[2,3,4,9,12] C. L. Chang,[12,2,3] H. C. Chiang,[2,13] H-M. Cho,[8,7] A. Conley,[7] T. M. Crawford,[2,4] T. de Haan,[1] M. A. Dobbs,[1] W. Everett,[7] J. Gallicchio,[2] J. Gao,[8] E. M. George,[14] N. W. Halverson,[7,15] N. Harrington,[14] J. W. Henning,[7] G. C. Hilton,[8] G. P. Holder,[1] W. L. Holzapfel,[14] J. D. Hrubes,[6] N. Huang,[14] J. Hubmayr,[8] K. D. Irwin,[8] R. Keisler,[2,9] L. Knox,[16] A. T. Lee,[14] E. Leitch,[2,4] D. Li,[8] C. Liang,[2,4] D. Luong-Van,[2] G. Marsden,[17] J. J. McMahon,[18] J. Mehl,[2,12] S. S. Meyer,[2,9,3,4] L. Mocanu,[2,4] T. E. Montroy,[19] T. Natoli,[2,9] J. P. Nibarger,[8] V. Novosad,[20] S. Padin,[10] C. Pryke,[21] C. L. Reichardt,[14] J. E. Ruhl,[19] B. R. Saliwanchik,[19] J. T. Sayre,[19] K. K. Schaffer,[2,22] B. Schulz,[10,23] G. Smecher,[1] A. A. Stark,[24] K. Story,[2,9] C. Tucker,[5] K. Vanderlinde,[1,25,26] J. D. Vieira,[10] M. P. Viero,[10] G. Wang,[12] V. Yefremenko,[12,20] O. Zahn,[27] and M. Zemcov[10,11]

B is from lensing of E
Can predict lensing potential from large-scale structure probe, e.g. CIB in this case
Measure E, make prediction for B by lensing E using CIB template for deflection prediction
Cross-correlate this prediction with actual B mode measurement
→ Agrees - detection!

Non-Gaussianity, statistical anisotropy
and reconstructing the lensing field

$\tilde{T}(\mathbf{x}) = T(\mathbf{x} + \nabla\psi)$ $P(T,\psi) \approx$ Gaussian; on small scales $\langle T\psi\rangle = 0 \Rightarrow P(T,\psi) = P(T)P(\psi)$

In pixels this is a remapping $\tilde{T}_i = [\Lambda(\psi)]_{ij}T_j$: Linear in T; non-linear in ψ

$$P(\tilde{T}) = \int \delta(\tilde{T} - \Lambda T)P(T,\psi)\mathrm{d}\psi\mathrm{d}T$$

1. Marginalize over (unobservable) lensing and unlensed temperature fields:

$\Rightarrow$ <u>Non-Gaussian statistically isotropic lensed temperature distribution</u>

2.

$$P(\tilde{T}) \approx \int P(\psi)\mathrm{d}\psi \underbrace{\int \delta(\tilde{T} - \Lambda T)P(T)\,\mathrm{d}T}_{P(\tilde{T}|\psi)}$$

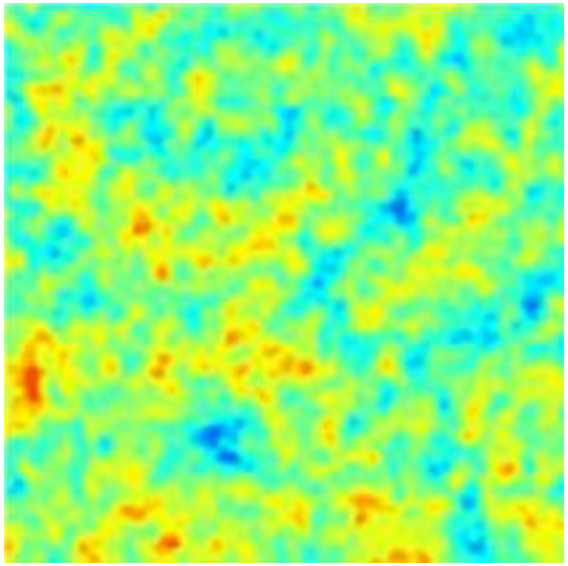
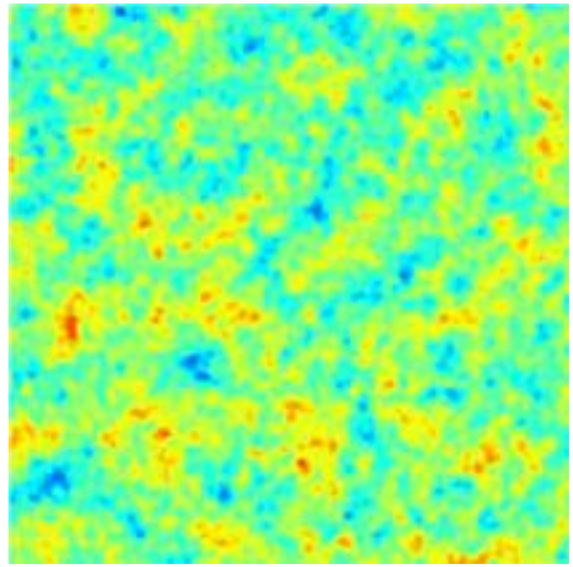
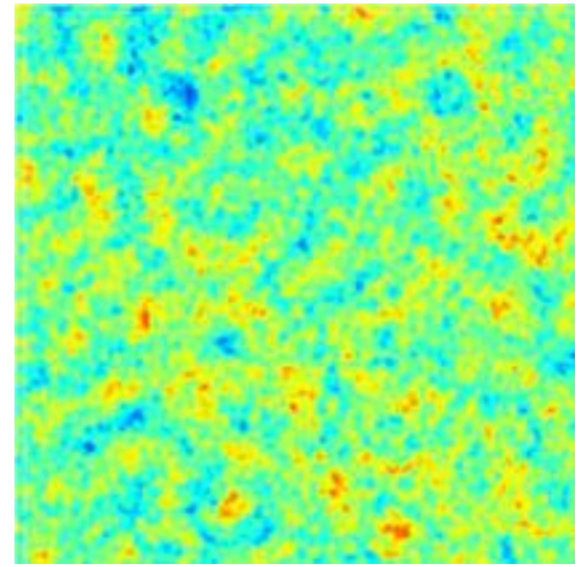
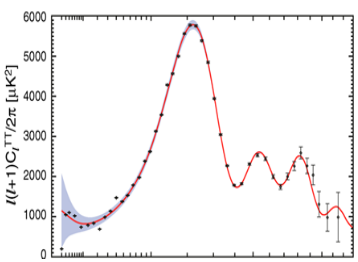
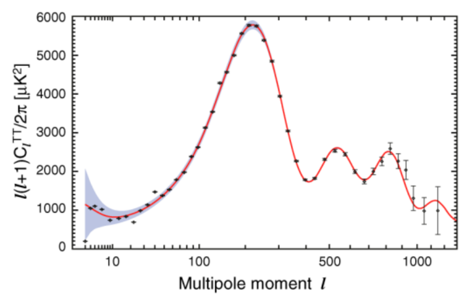
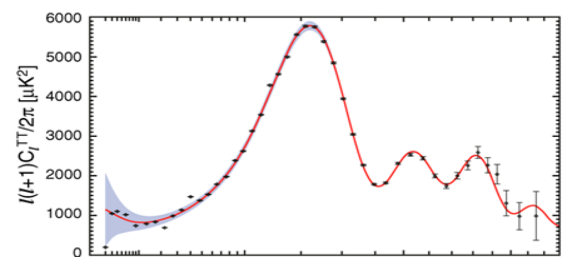
$$\Rightarrow \quad P(\tilde{T}) = \int P(\tilde{T}|\psi)P(\psi)\mathrm{d}\psi$$

For a given lensing field think about $\tilde{T} \sim P(\tilde{T}|\psi)$:

$\tilde{T} = \Lambda T$ is a *linear* function of T for fixed ψ:

$\Rightarrow$ <u>Anisotropic Gaussian lensed temperature distribution</u>

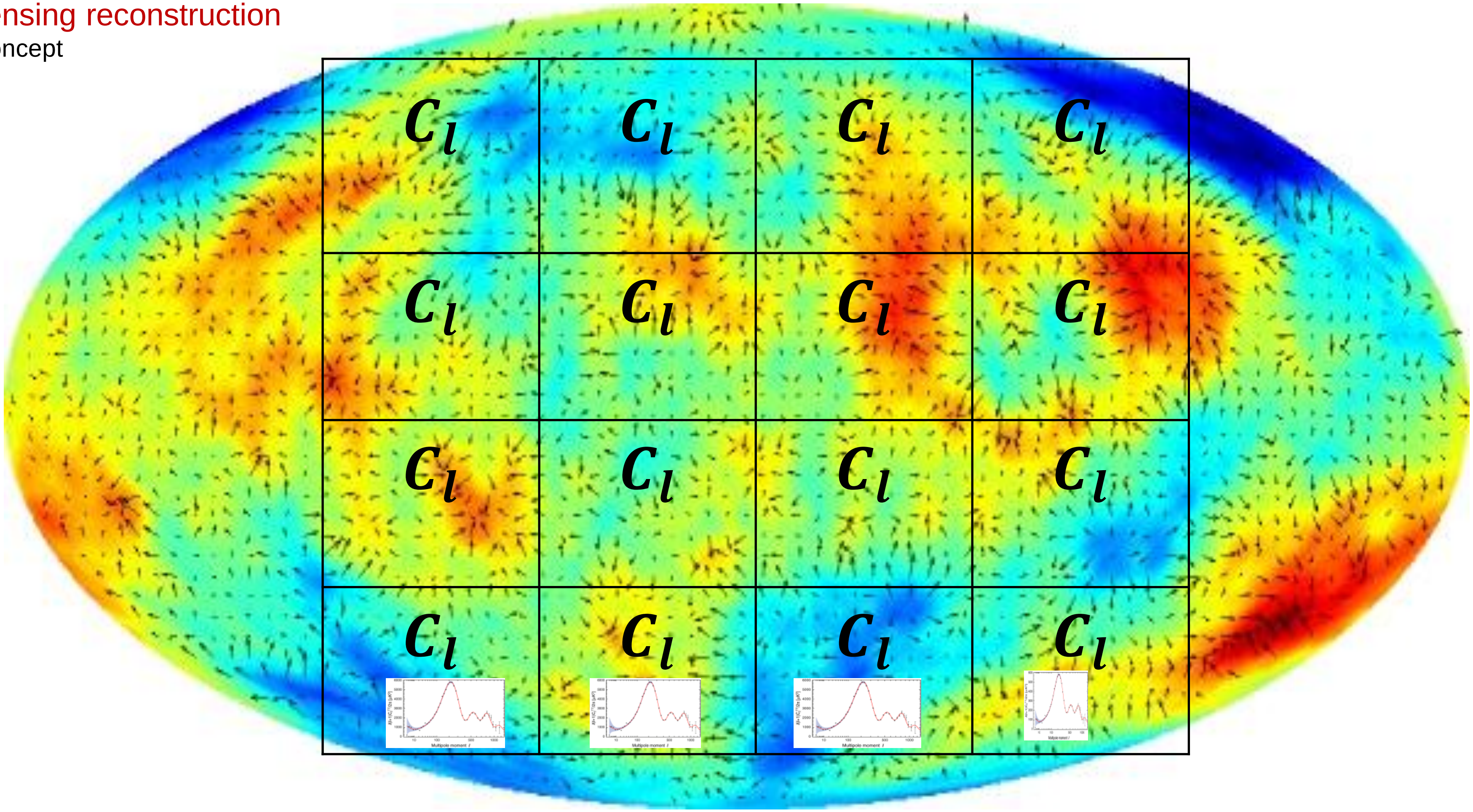
Think about ‘squeezed’ configuration: big nearly constant lenses, much smaller lensed T

Magnified	Unlensed	Demagnified
		
		

Fractional magnification ~ convergence $\kappa = -\boldsymbol{\nabla} \cdot \frac{\boldsymbol{\alpha}}{2}$ + shear modulation:

Lensing reconstruction

-concept



Variance in each C_l measurement $\propto 1/N_{\text{modes}}$

$N_{\text{modes}} \propto l_{\text{max}}^2$ - dominated by smallest scales

$\Rightarrow$ measurement of angular scale ($\Rightarrow \kappa$) in each box nearly independent
$\Rightarrow$ Uncorrelated variance on estimate of magnificantion κ in each box
$\Rightarrow$ Nearly white 'reconstruction noise' $N_l^{(0)}$ on κ , with $N_l^{(0)} \propto 1/l_{\text{max}}^2$

For fixed ψ: Gaussian anisotropic distribution $\Rightarrow \langle\Theta(\mathbf{l})\Theta(\mathbf{l}')\rangle \neq C_l \delta(\mathbf{l}-\mathbf{l}')$

Use series expansion:

$$\tilde{\Theta}(\mathbf{l}) \approx \Theta(\mathbf{l}) - \int \frac{\mathrm{d}^2\mathbf{l}'}{2\pi}\, \mathbf{l}' \cdot (\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}')\Theta(\mathbf{l}')$$

(higher order terms are important, but bias can be corrected for later)

Average over unlensed CMB Θ:

$$\langle\tilde{\Theta}(\mathbf{l})\tilde{\Theta}^*(\mathbf{l}-\mathbf{L})\rangle_\Theta = \delta(\mathbf{L})\, C_l^\Theta + \frac{1}{2\pi}\left[(\mathbf{L}-\mathbf{l})\cdot\mathbf{L}\, C^\Theta_{|\mathbf{l}-\mathbf{L}|} + \mathbf{l}\cdot\mathbf{L}\, C_l^\Theta\right]\psi(\mathbf{L}) + \mathcal{O}(\psi^2)$$

Off-diagonal correlation $\propto \psi(L)$ – use to measure ψ!

For $L \geq 1$ define quadratic estimator by summing up with weights $g(\boldsymbol{l}, \boldsymbol{L})$

$$\hat{\psi}(\mathbf{L}) \equiv N(\mathbf{L}) \int \frac{\mathrm{d}^2\mathbf{l}}{2\pi}\, \tilde{\Theta}(\mathbf{l})\tilde{\Theta}^*(\mathbf{l}-\mathbf{L})g(\mathbf{l},\mathbf{L}).$$

Zaldarriaga & Seljak, Hu 2001+

Want $\langle\hat{\psi}(\mathbf{L})\rangle_{\Theta} = \psi(\mathbf{L})$

$$\Rightarrow \quad N(\mathbf{L})^{-1} = \int \frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\left[(\mathbf{L}-\mathbf{l})\cdot\mathbf{L}\,C^{\Theta}_{|\mathbf{l}-\mathbf{L}|} + \mathbf{l}\cdot\mathbf{L}\,C^{\Theta}_{l}\right] g(\mathbf{l},\mathbf{L})$$

Want the best estimator: find weights g to minimize the variance

$$\langle\hat{\psi}^*(\mathbf{L})\hat{\psi}(\mathbf{L}')\rangle = \delta(\mathbf{L}-\mathbf{L}')2N(\mathbf{L})^2\int\frac{\mathrm{d}^2\mathbf{l}}{(2\pi)^2}\tilde{C}^{\mathrm{tot}}_{l}\tilde{C}^{\mathrm{tot}}_{|\mathbf{l}-\mathbf{L}|}[g(\mathbf{l},\mathbf{L})]^2 + \mathcal{O}(C^{\psi}_{l})$$

$$\tilde{C}^{\mathrm{tot}}_{l} = \tilde{C}^{\Theta}_{l} + N_l$$

$$g(\mathbf{l},\mathbf{L}) = \frac{(\mathbf{L}-\mathbf{l})\cdot\mathbf{L}\,C^{\Theta}_{|\mathbf{l}-\mathbf{L}|} + \mathbf{l}\cdot\mathbf{L}\,C^{\Theta}_{l}}{2\tilde{C}^{\mathrm{tot}}_{l}\tilde{C}^{\mathrm{tot}}_{|\mathbf{l}-\mathbf{L}|}}$$

Reconstruction 'Noise' – from random fluctuations of the unlensed CMB
Turns out to be the same as the normalization $N(L)$

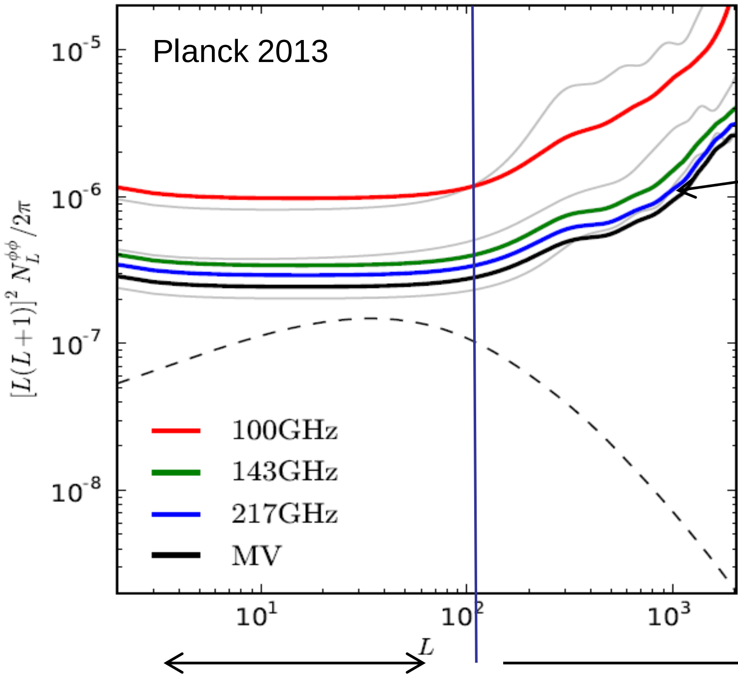
$$\delta(\mathbf{0})\langle|\hat{\psi}(\mathbf{L})|^2\rangle^{-1} = N(\mathbf{L})^{-1} = \int \frac{d^2\mathbf{l}}{(2\pi)^2} \frac{\left[(\mathbf{L}-\mathbf{l})\cdot\mathbf{L}\,C^{\Theta}_{|\mathbf{l}-\mathbf{L}|} + \mathbf{l}\cdot\mathbf{L}\,C^{\Theta}_{l}\right]^2}{2\tilde{C}^{\rm tot}_{l}\tilde{C}^{\rm tot}_{|\mathbf{l}-\mathbf{L}|}}$$

(in limit of no lensing – there are higher order corrections)

On large scales (large lenses), $L \ll l$, with no instrumental noise

$$\frac{1}{L^4 N(L)} \approx \frac{1}{16\pi}\int l\,dl \left(\underbrace{\left[\frac{d\ln l^2 C_l}{d\ln l}\right]^2}_{\text{Convergence}} + \underbrace{\frac{1}{2}\left[\frac{d\ln C_l}{d\ln l}\right]^2}_{\text{Shear}} \right)$$ constant

$\propto C_l^{\kappa}$



Reconstruction Noise
$N_l^{(0)}$

$N_l^{(0)} \sim$ const

Beam, noise, shape of C_l and $l \sim l_{\max}$ effects

Lensing reconstruction information mostly in the *smallest scales* observed

- Need high resolution and sensitivity
- Almost totally insensitive to large-scale T (so only *small-scale* foregrounds an issue)

Practical fast way to do it, using FFT:

$$\hat{\psi}(\mathbf{L}) = N(\mathbf{L})\,\mathbf{L}\cdot\int \frac{d^2\mathbf{l}}{2\pi}\,\frac{\mathbf{l}C_l^{\Theta}\tilde{\Theta}(\mathbf{l})}{\tilde{C}_l^{\text{tot}}}\,\frac{\tilde{\Theta}(\mathbf{L}-\mathbf{l})}{\tilde{C}_{|\mathbf{L}-\mathbf{l}|}^{\text{tot}}}$$

- Looks like convolution: use convolution theorem

$$\begin{aligned}\hat{\psi}(\mathbf{L}) &= -iN(\mathbf{L})\,\mathbf{L}\cdot\int \frac{d^2\mathbf{x}}{2\pi}\,e^{-i\mathbf{L}\cdot\mathbf{x}}F_1(\mathbf{x})\nabla F_2(\mathbf{x}) \\ &= -N(\mathbf{L})\int \frac{d^2\mathbf{x}}{2\pi}\,e^{-i\mathbf{L}\cdot\mathbf{x}}\nabla\cdot\underbrace{[F_1(\mathbf{x})\nabla F_2(\mathbf{x})]}_{\text{Easy to calculate in real space: multiply maps}}\end{aligned}$$

$$F_1(\mathbf{l}) \equiv \frac{\tilde{\Theta}(\mathbf{l})}{\tilde{C}_l^{\text{tot}}} \qquad F_2(\mathbf{l}) \equiv \frac{\tilde{\Theta}(\mathbf{l})C_l^{\Theta}}{\tilde{C}_l^{\text{tot}}}$$

- fast and easy to compute in harmonic space

- Can make similar argument on full sky and for polarization

Alternative more general derivation (works for cut-sky, anisotropic noise)
For fixed lenses, sky is Gaussian by anisotropic:

$$-\mathscr{L}(\hat{\Theta}|\alpha) = \frac{1}{2}\hat{\Theta}^T \left(\hat{C}^{\hat{\Theta}\hat{\Theta}}\right)^{-1}\hat{\Theta} + \frac{1}{2}\ln\det(\hat{C}^{\hat{\Theta}\hat{\Theta}}),$$

Find the maximum-likelihood estimator for the lensing potential/deflection angle

$$\frac{\delta\mathscr{L}}{\delta\alpha_i(\mathbf{x})} = \frac{1}{2}\hat{\Theta}^T(\hat{C}^{\hat{\Theta}\hat{\Theta}})^{-1}\frac{\delta\hat{C}^{\hat{\Theta}\hat{\Theta}}}{\delta\alpha_i(\mathbf{x})}(\hat{C}^{\hat{\Theta}\hat{\Theta}})^{-1}\hat{\Theta} - \frac{1}{2}\mathrm{Tr}\left[(\hat{C}^{\hat{\Theta}\hat{\Theta}})^{-1}\frac{\delta\hat{C}^{\hat{\Theta}\hat{\Theta}}}{\delta\alpha_i(\mathbf{x})}\right] = 0$$

Trick: $Tr(A) = Tr(AC^{-1}\langle xx^T\rangle)$ where x has covariance $\boldsymbol{C}$

$= \langle x^T AC^{-1}x\rangle$ - rewrite trace as "mean field" average

Can show that at leading order maximum likelihood solution is as before, but with

$$\frac{1}{C_l^{tot}} \to (\hat{C}^{\hat{\Theta}\hat{\Theta}})^{-1} = (S+N)^{-1}$$

Sets to zero in cut, downweights high noise (also need *lensed* C_l)

$$\hat{\psi} \to \hat{\psi} - \langle\hat{\psi}\rangle$$

weights optimally for cuts/noise and subtracts average signal from noise inhomogeneity and cuts

"mean field" calculated from simulations

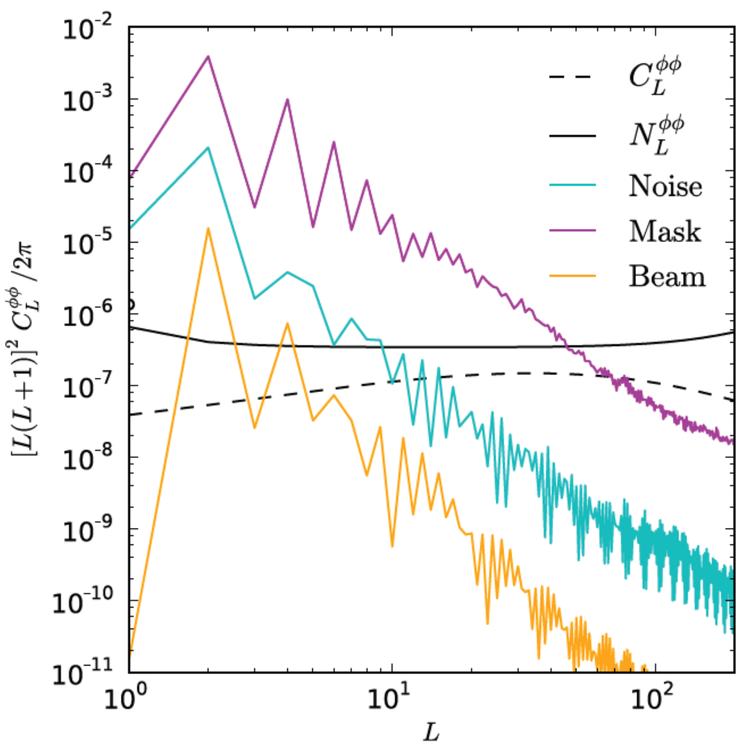
astro-ph/0209489, arXiv:0908.0963

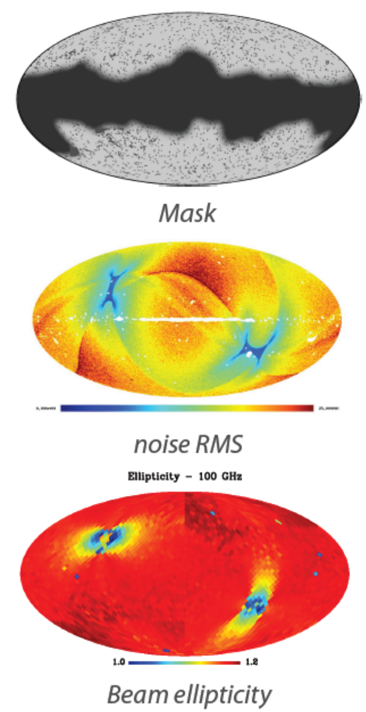
Final “optimal” quadratic (QML) estimator $\hat{\psi}(K)$

$$\hat{\psi}(L) \sim N_L^{(0)}\left[\sum_{\mathbf{k}_2,\mathbf{k}_3} A(L,k_2,k_3)\,\bar{\Theta}(\mathbf{k}_2)\bar{\Theta}(\mathbf{k}_3) - (\text{Monte carlo for zero signal})\right]$$

Filtered maps $\bar{\Theta} = (S+N)^{-1}\Theta$
weights for mask and noise anisotropy
(here approx. noise term as diagonal)

“Mean field” – accounts for other sources of anisotropy in the data



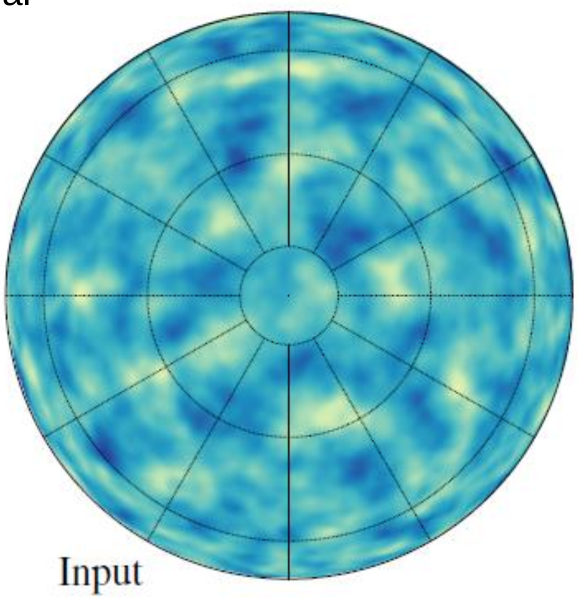


- “Mean-field” corrections are very large at low-L. We fail some detailed consistency tests at $L < 10$ (though not very badly!).

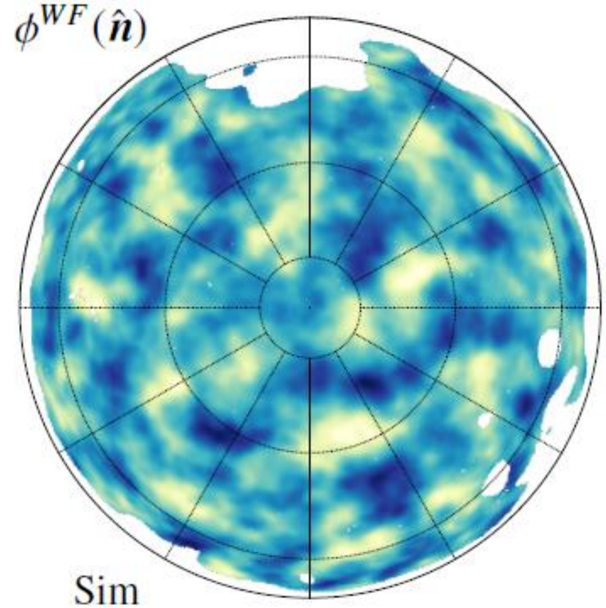
Planck 2013

Planck simulation

True lensing potential

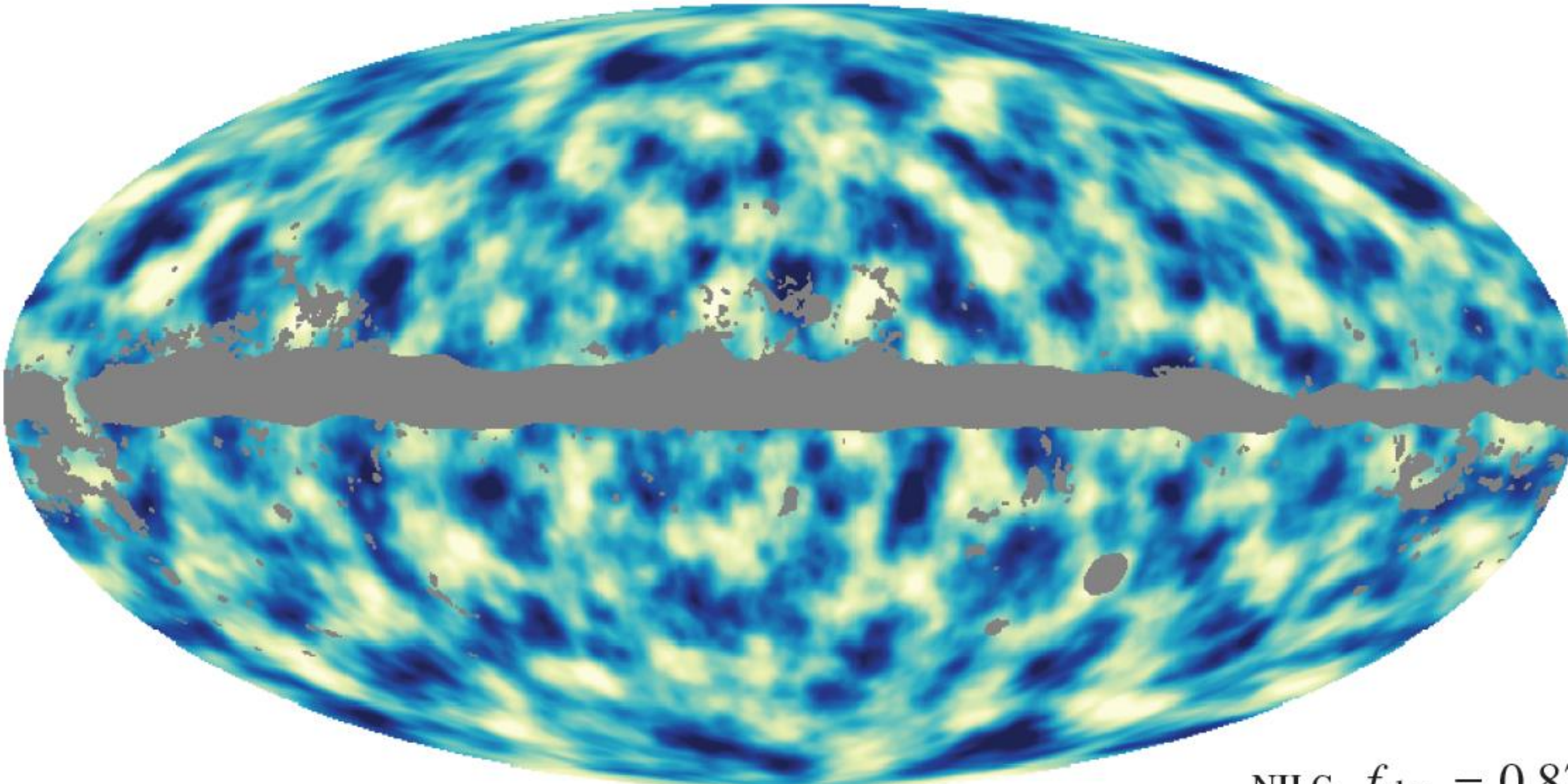


S/(S+N) filtered reconstruction



= input + ‘reconstruction noise’

Planck full-sky lensing potential reconstruction

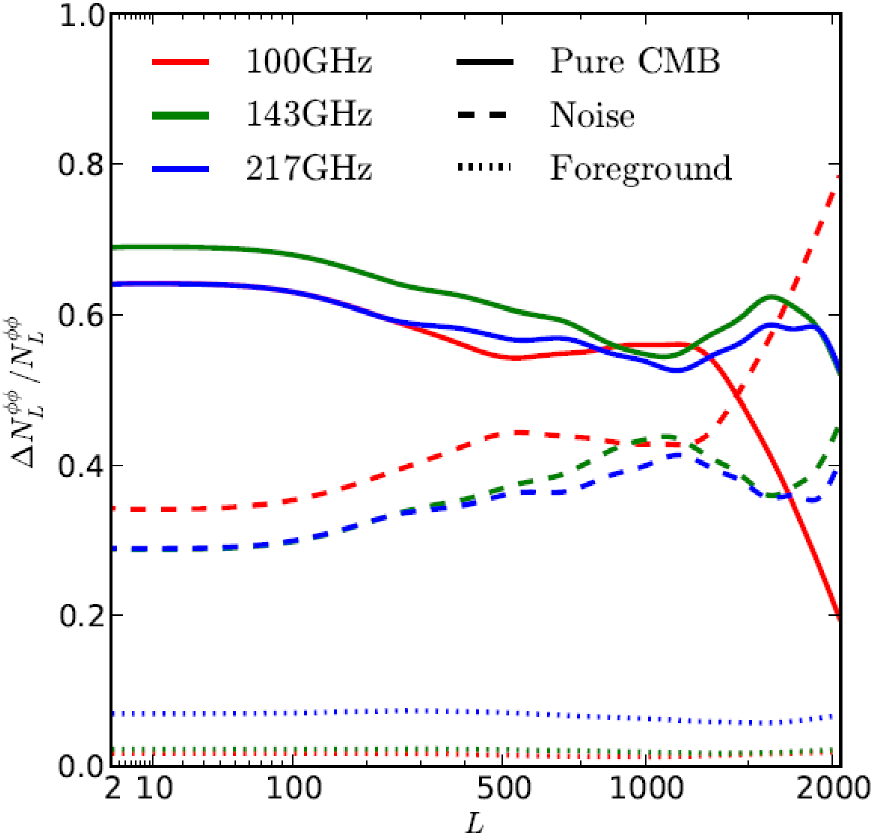


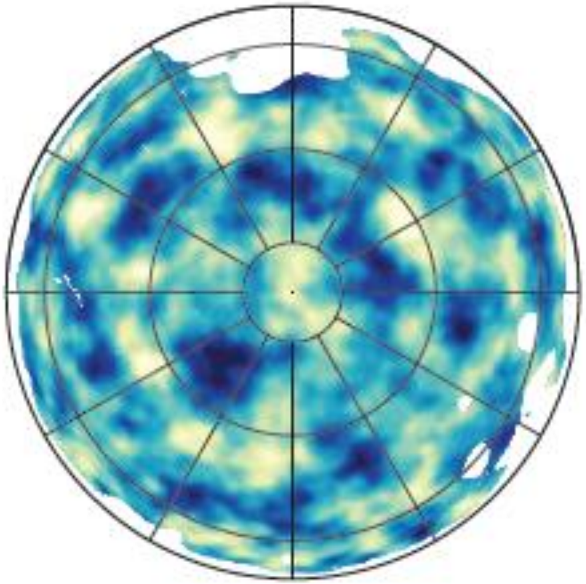
NILC, $f_{\rm sky} = 0.87$

Note – about half signal, half noise,
not all structures are real: map is effectively Wiener filtered

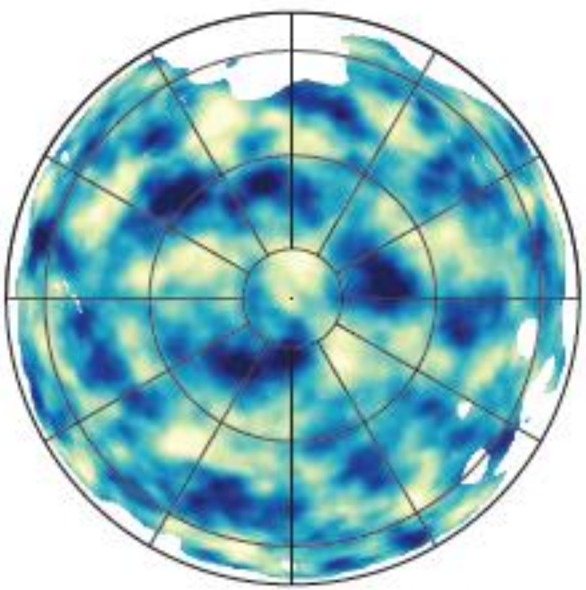
Reconstruction noise budget

Lensing maps are reconstruction noise dominated, but maps from different channels are similar because mainly the same CMB cosmic variance.



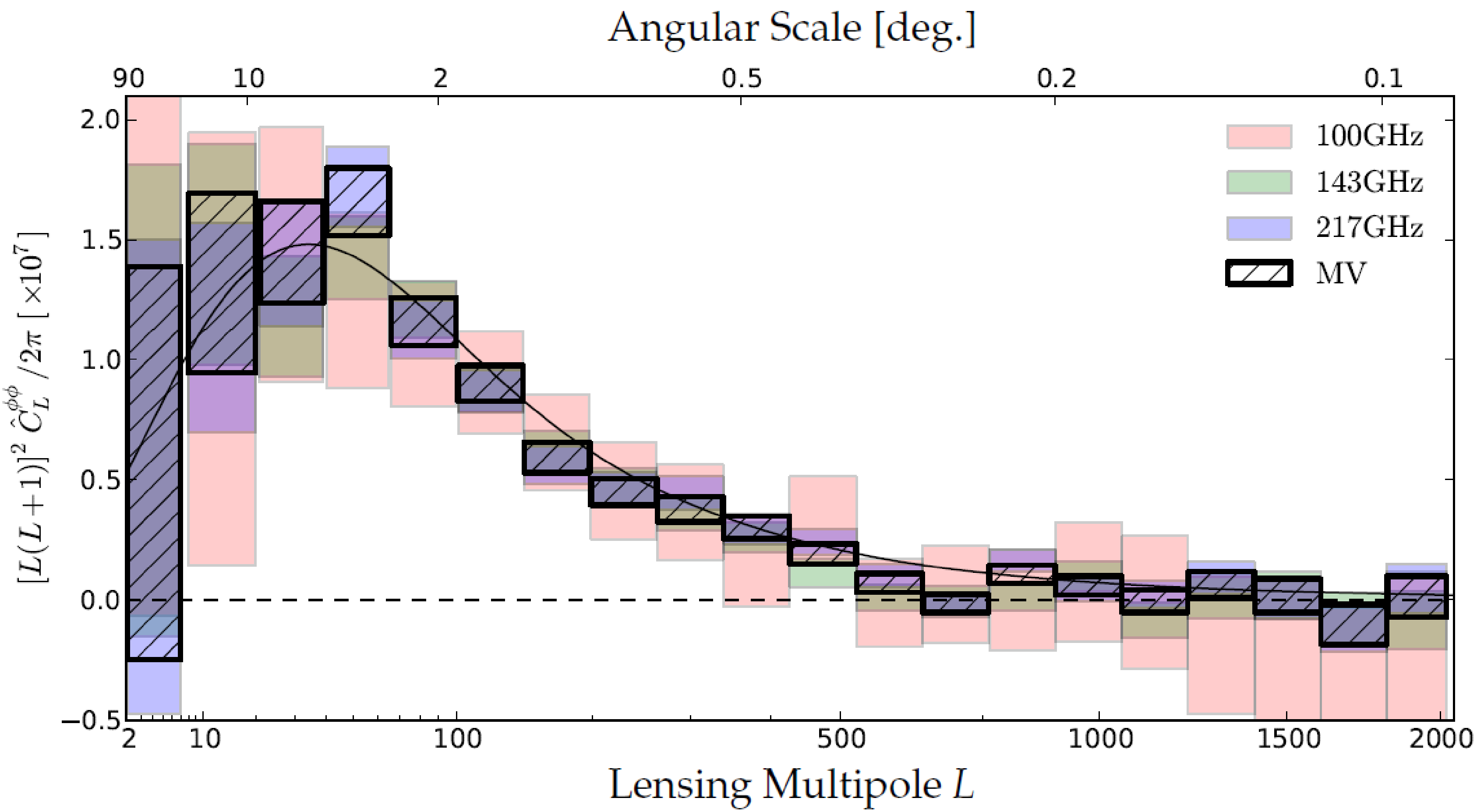


Galactic South - 143 GHz



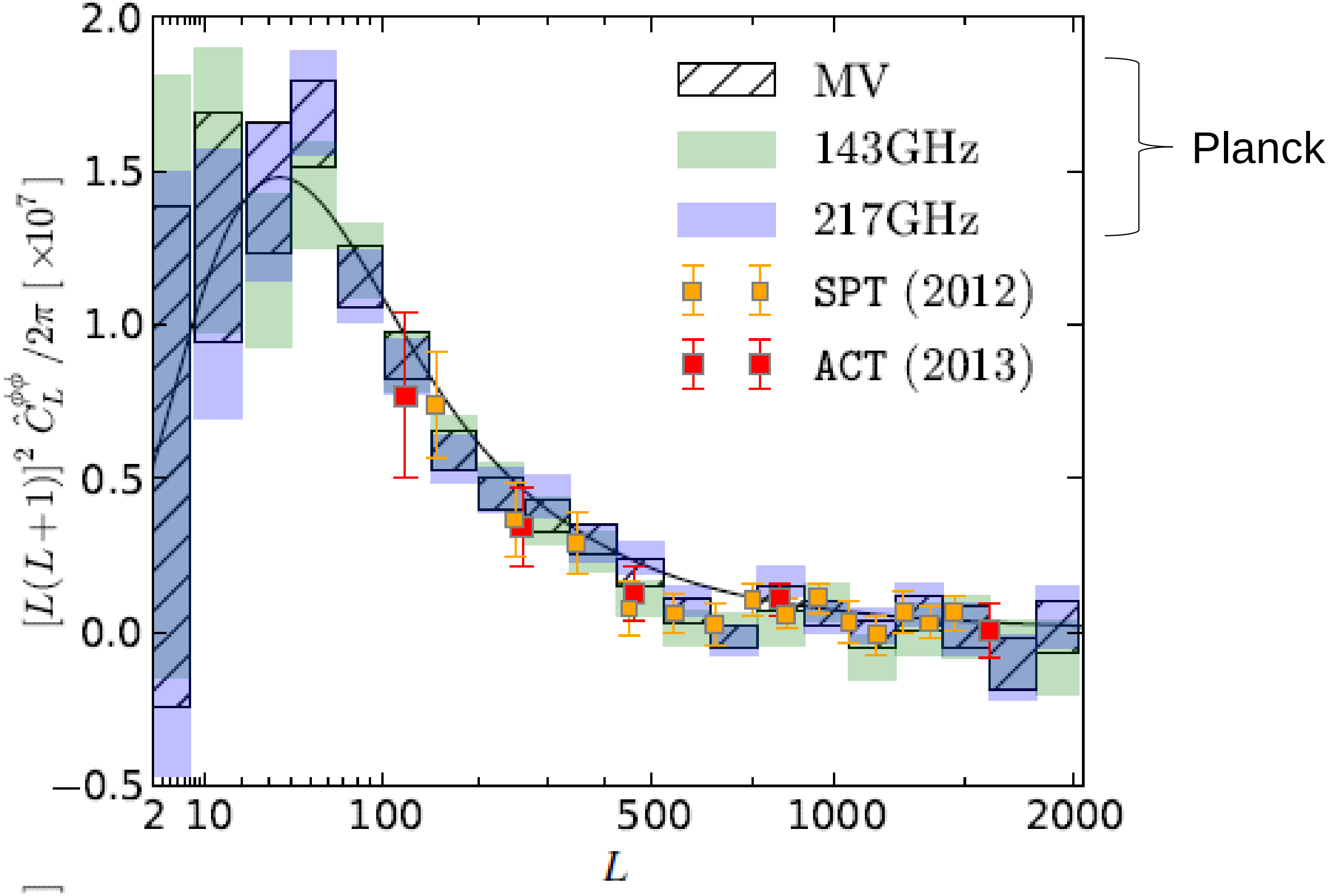
Galactic South - 217 GHz

Power spectrum of reconstruction ⇒ $C_l^{\psi\psi}$



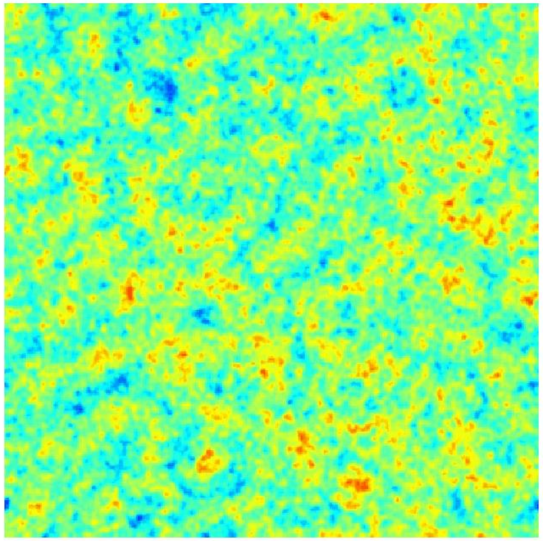
[Also need to subtract next order power spectrum biases, $N^{(1)}$]

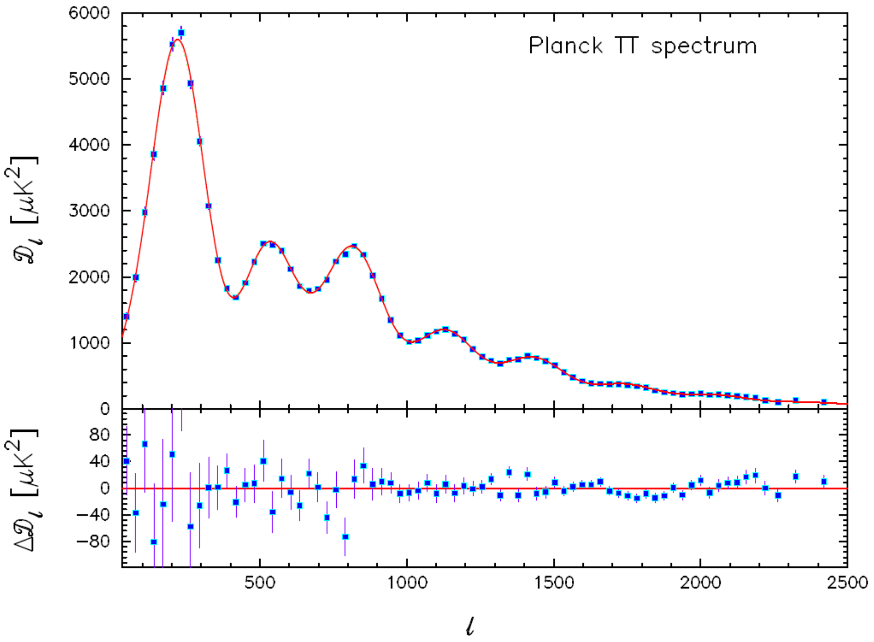
Comparison with ACT/SPT



(new SPT data also coming “soon”)

Is it useful for parameter estimation?





Detailed measurement of 6 power spectrum acoustic peaks in TT

Accurate measurement of cosmological parameters?

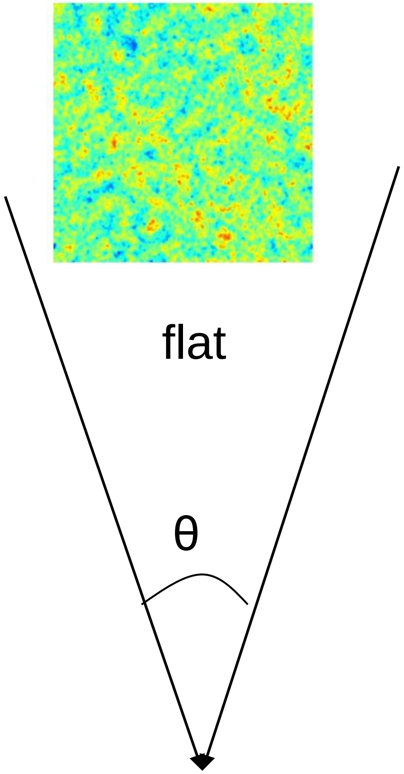
YES: some particular parameters measured very accurately

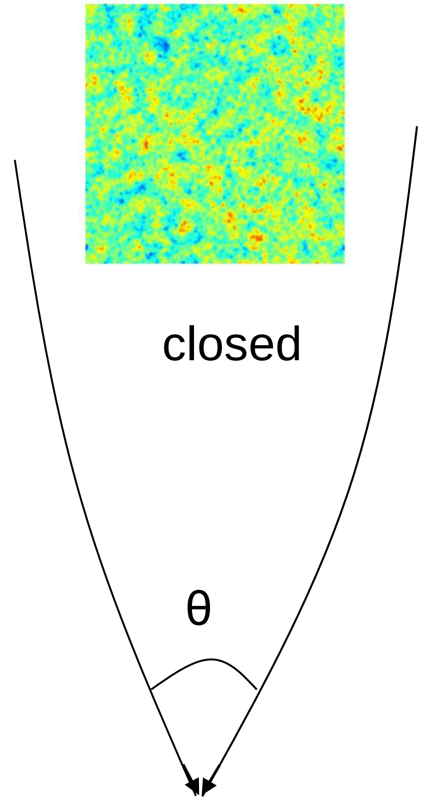
0.1% accurate measurement of the acoustic scale:

$$\theta_* = (1.04148 \pm 0.00066) \times 10^{-2} = 0.596724^\circ \pm 0.00038^\circ.$$

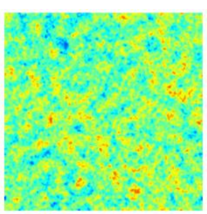
But need full cosmological model to relate to underlying physical parameters..

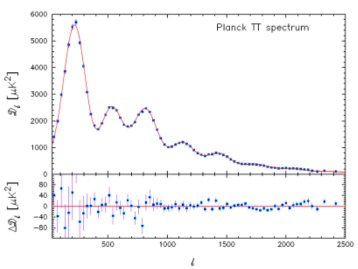
e.g. Geometry: curvature

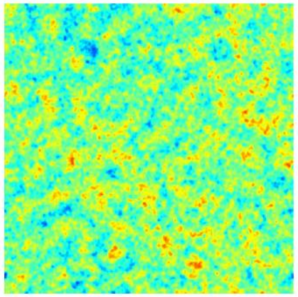


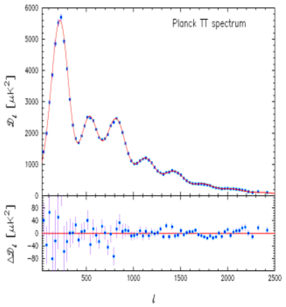


We see:

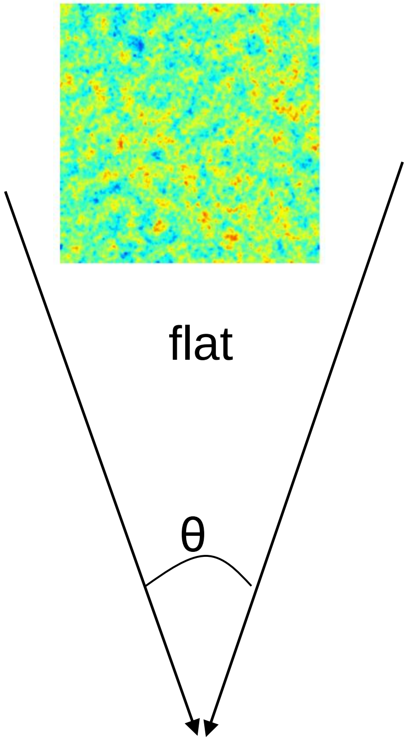


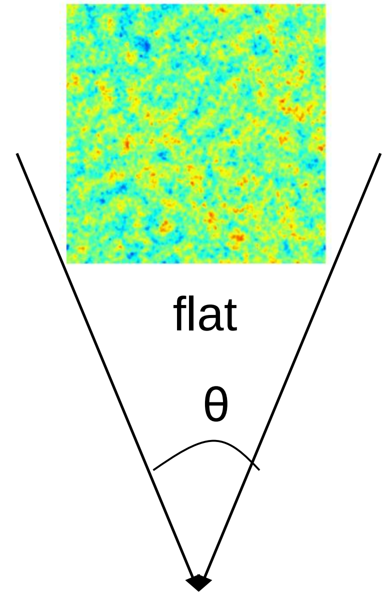




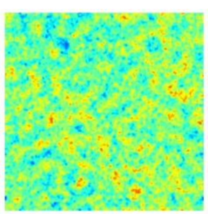


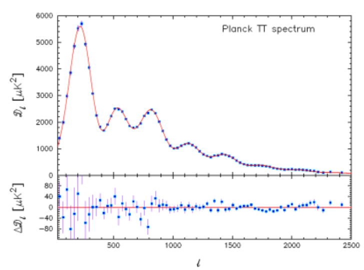
or is it just closer??

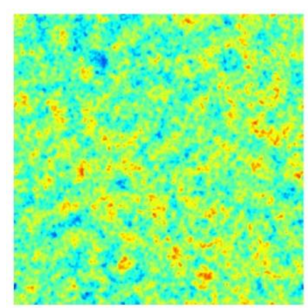


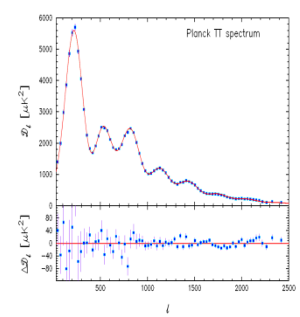


We see:





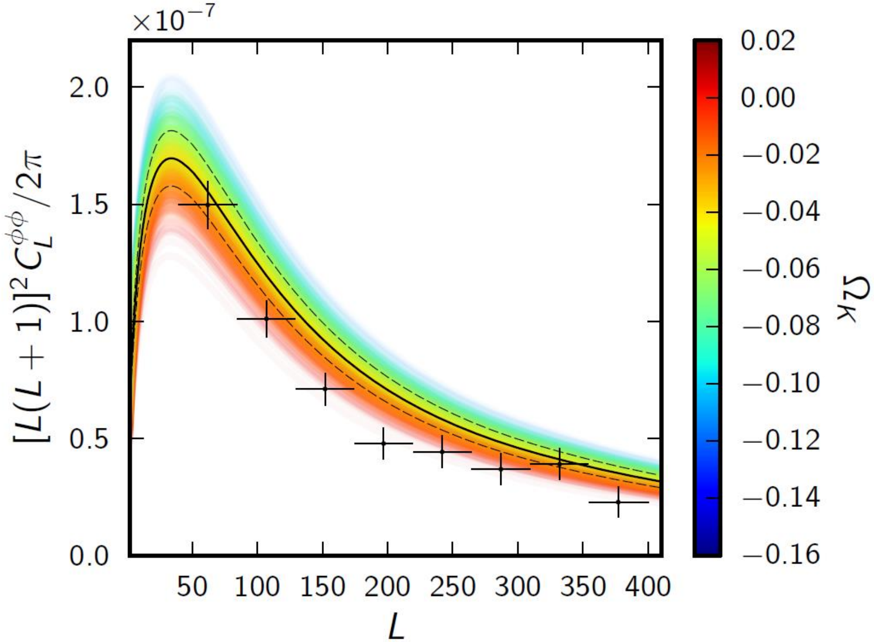


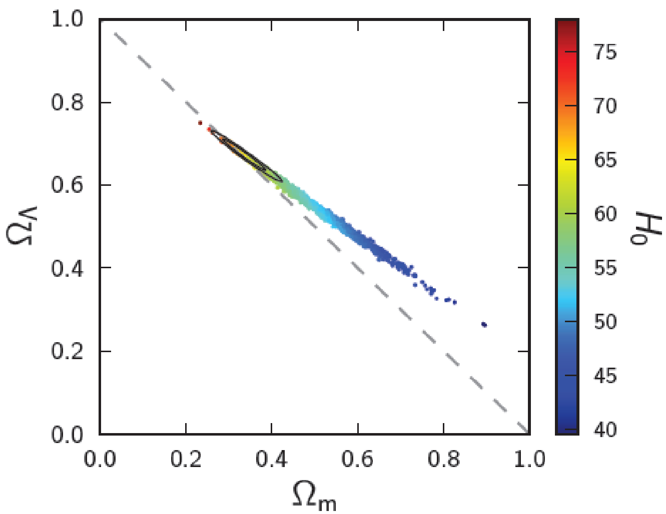


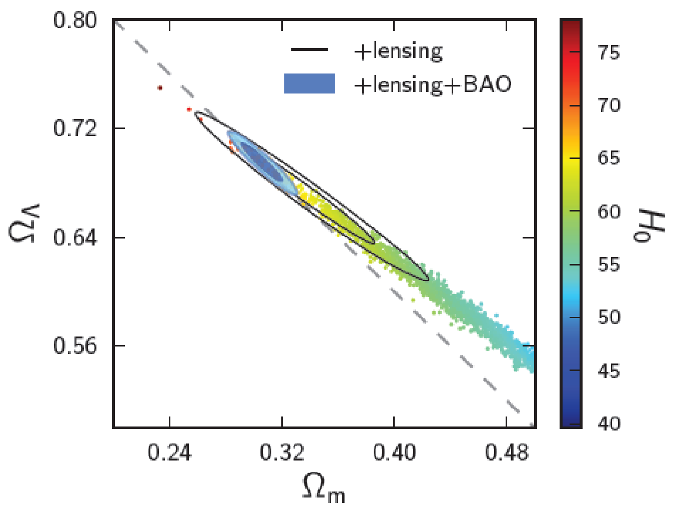
Degeneracies between parameters

Extra lensing information can help break parameter degeneracies

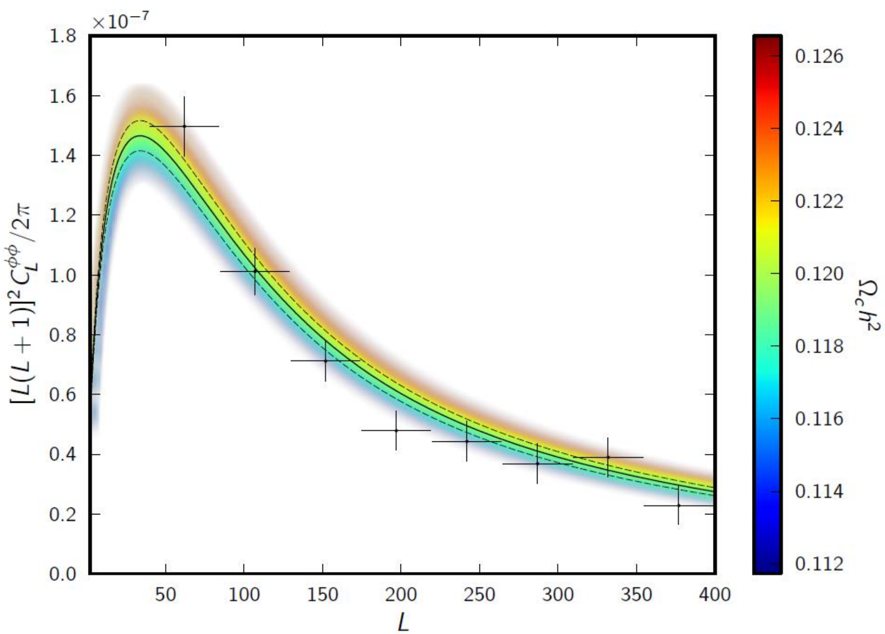
Colour: Planck TT constraint
Crosses: Planck lensing



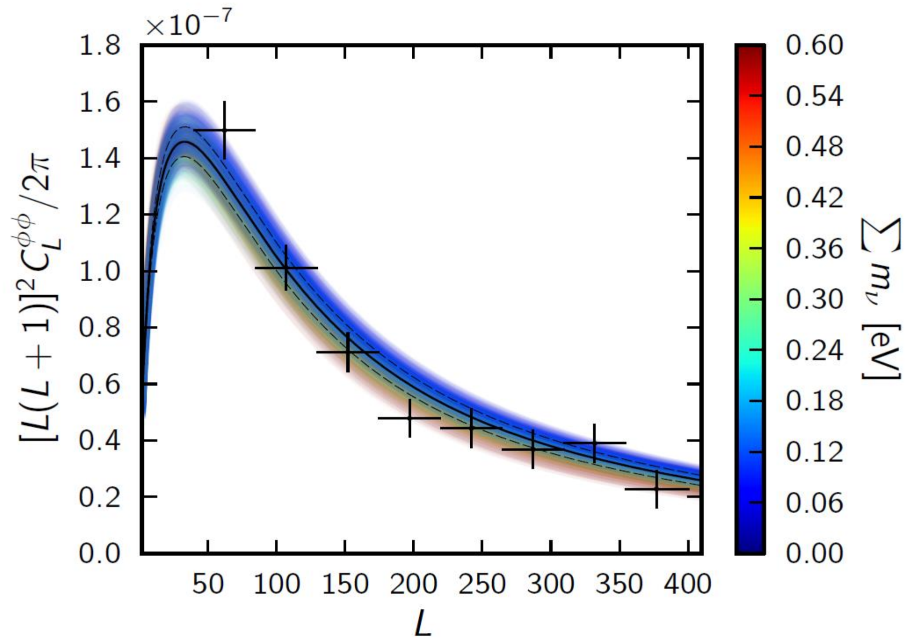




LCDM matter density



Neutrino mass

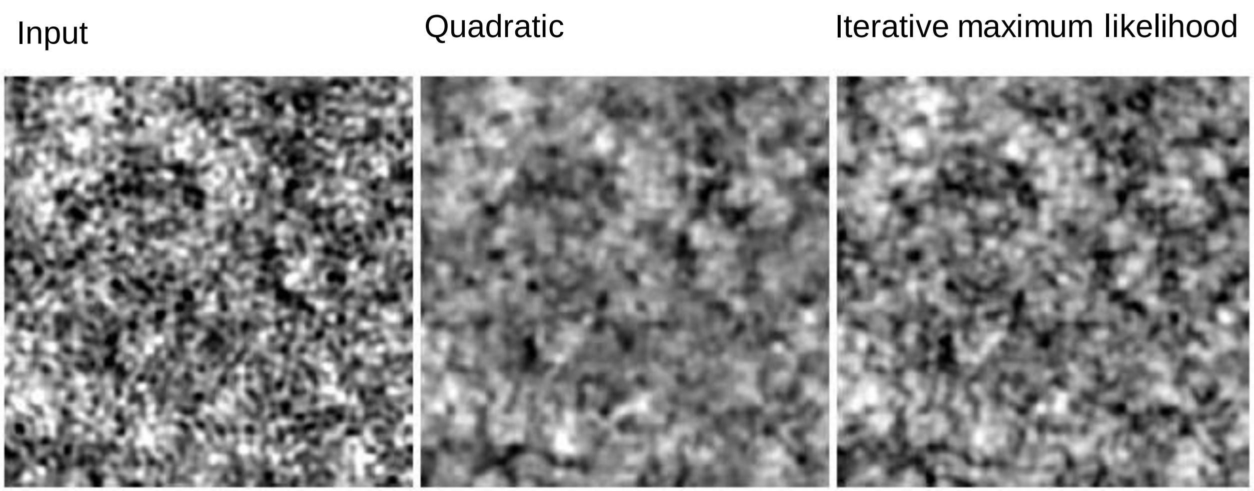


Polarization lensing reconstruction

- Reconstruction with polarization is **much better** if the noise is low enough: no cosmic variance in unlensed B – all small-scale B is lensing

- Polarization reconstruction can in principle be used to de-lens the CMB
  - required to probe tensor amplitudes r <~ 10^{-4}
  - requires very high sensitivity and high resolution
  - in principle can do things almost exactly: a lot of information in lensed B at high l

- Maximum likelihood techniques much better than quadratic estimators for polarization (astro-ph/0306354)



- Lensed CMB power spectra contain essentially two new numbers:
  - one from T and E, depends on lensing potential at $\boldsymbol{l}<300$
  - one from lensed BB, wider range of $\boldsymbol{l}$

  astro-ph/0607315

- More information can be obtained from lensing reconstruction

- Correlation between reconstruction and power spectrum lensing currently negligible because of large reconstruction noise; in future may have to model more carefully

Non-Gaussianity

Flat sky approximation: $\Theta(x) = \frac{1}{2\pi}\int d^2 l\, \Theta(l) e^{ix\cdot l}$ $(\Theta = T)$

If no non-Gaussianity: Gaussian + statistical isotropy:

$$\langle \Theta(l_1)\Theta(l_2)\rangle = \delta(l_1 + l_2) C_l$$

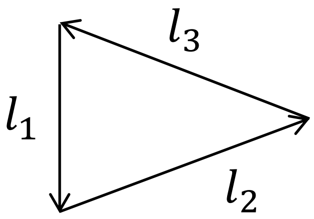
- power spectrum encodes all the information

- modes with different wavenumber are independent

For more details on the following see arXiv:1101.2234, 1107.5431 and refs therein

Non-Gaussianity – general possibilities

Bispectrum



$$\boldsymbol{l_1} + \boldsymbol{l_2} + \boldsymbol{l_3} = \boldsymbol{0}$$

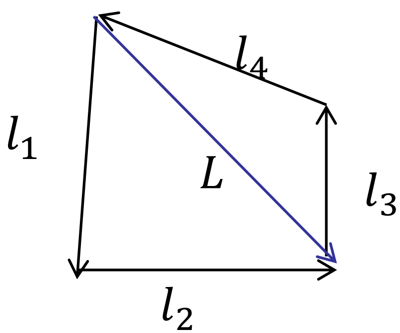
Flat sky approximation: $\langle \Theta(l_1)\Theta(l_2)\Theta(l_3)\rangle = \frac{1}{2\pi}\delta(l_1 + l_2 + l_3) b_{l_1 l_2 l_3}$

If you know $\Theta(l_1), \Theta(l_2)$, sign of $b_{l_1 l_2 l_3}$ tells you which sign of $\Theta(l_3)$ is more likely

Trispectrum

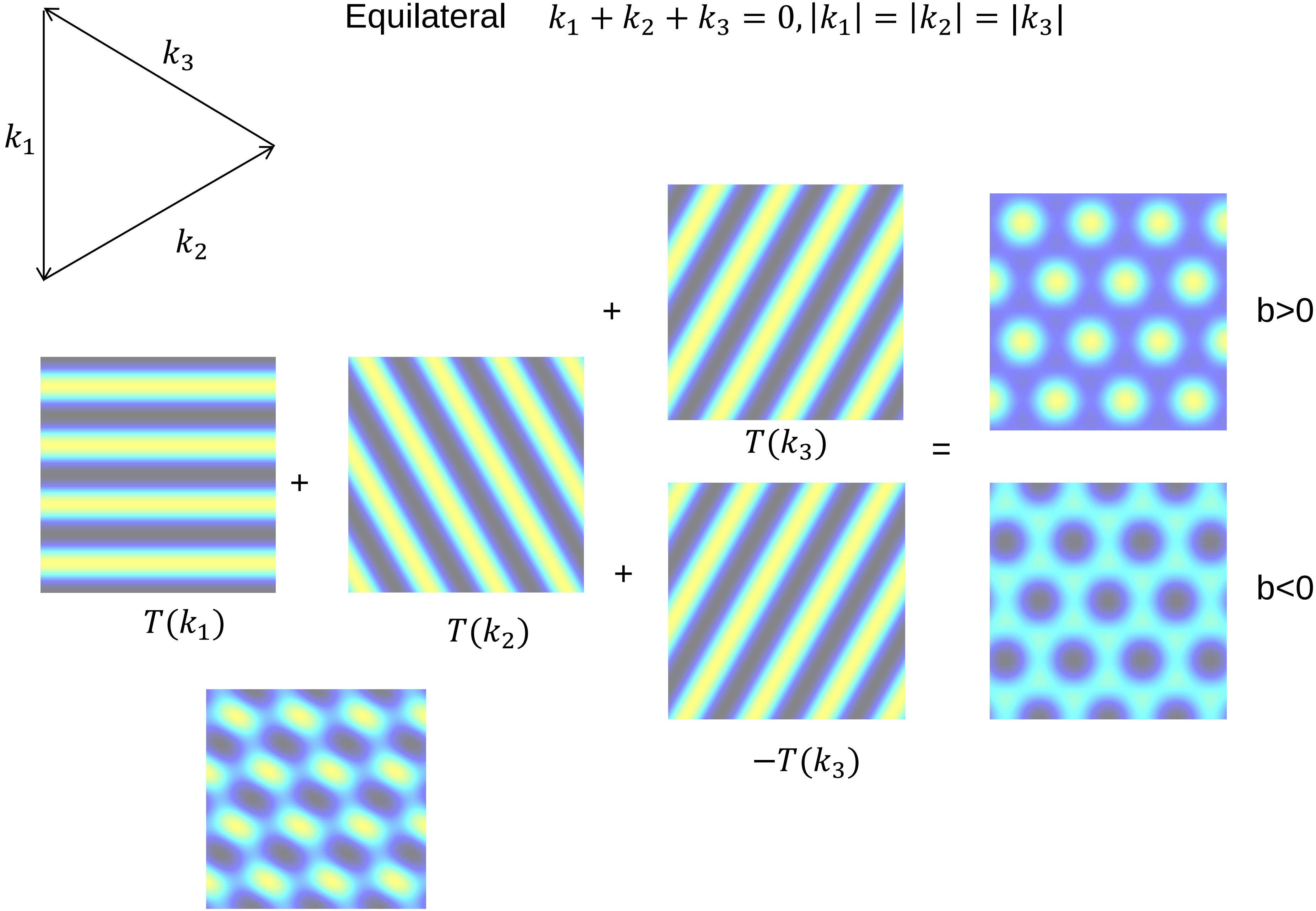
$$\langle \Theta(\mathbf{l}_1)\Theta(\mathbf{l}_2)\Theta(\mathbf{l}_3)\Theta(\mathbf{l}_4)\rangle_C = (2\pi)^{-2}\delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 + \mathbf{l}_4) T(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4)$$

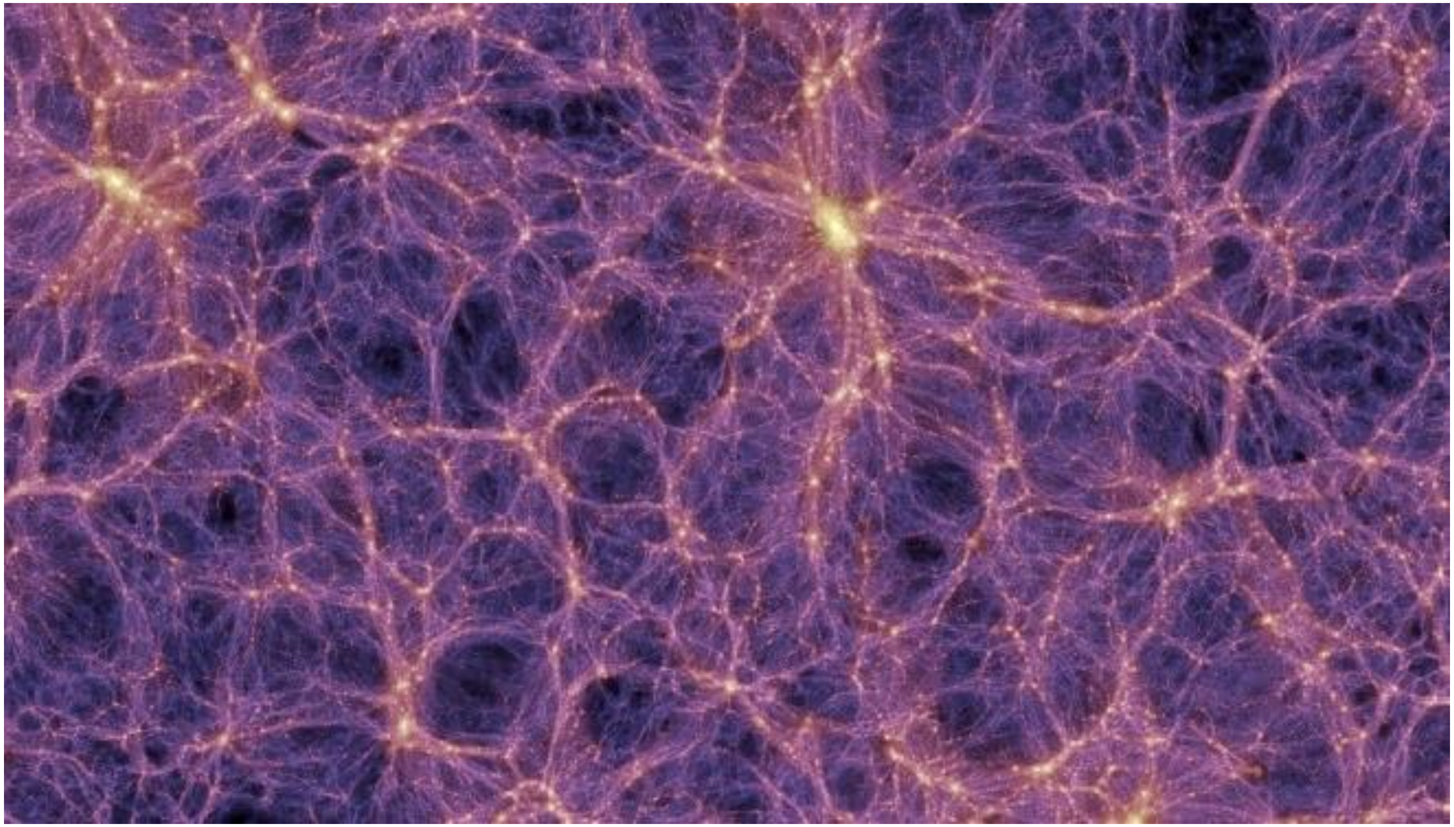
$$\langle \Theta(\mathbf{l}_1)\Theta(\mathbf{l}_2)\Theta(\mathbf{l}_3)\Theta(\mathbf{l}_4)\rangle_C = \frac{1}{2}\int \frac{d^2\mathbf{L}}{(2\pi)^2}\delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{L})\delta(\mathbf{l}_3 + \mathbf{l}_4 - \mathbf{L})\mathbb{T}^{(\ell_1\ell_2)}_{(\ell_3\ell_4)}(L) + \text{perms.}$$



N-spectra…

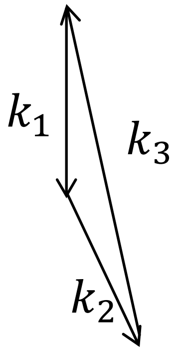
Equilateral $k_1 + k_2 + k_3 = 0, |k_1| = |k_2| = |k_3|$

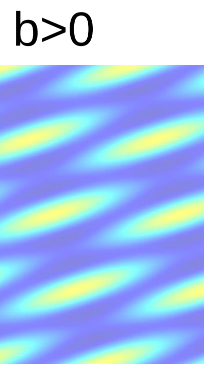


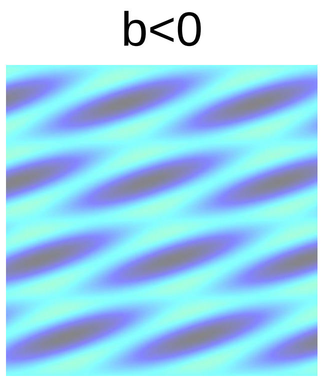


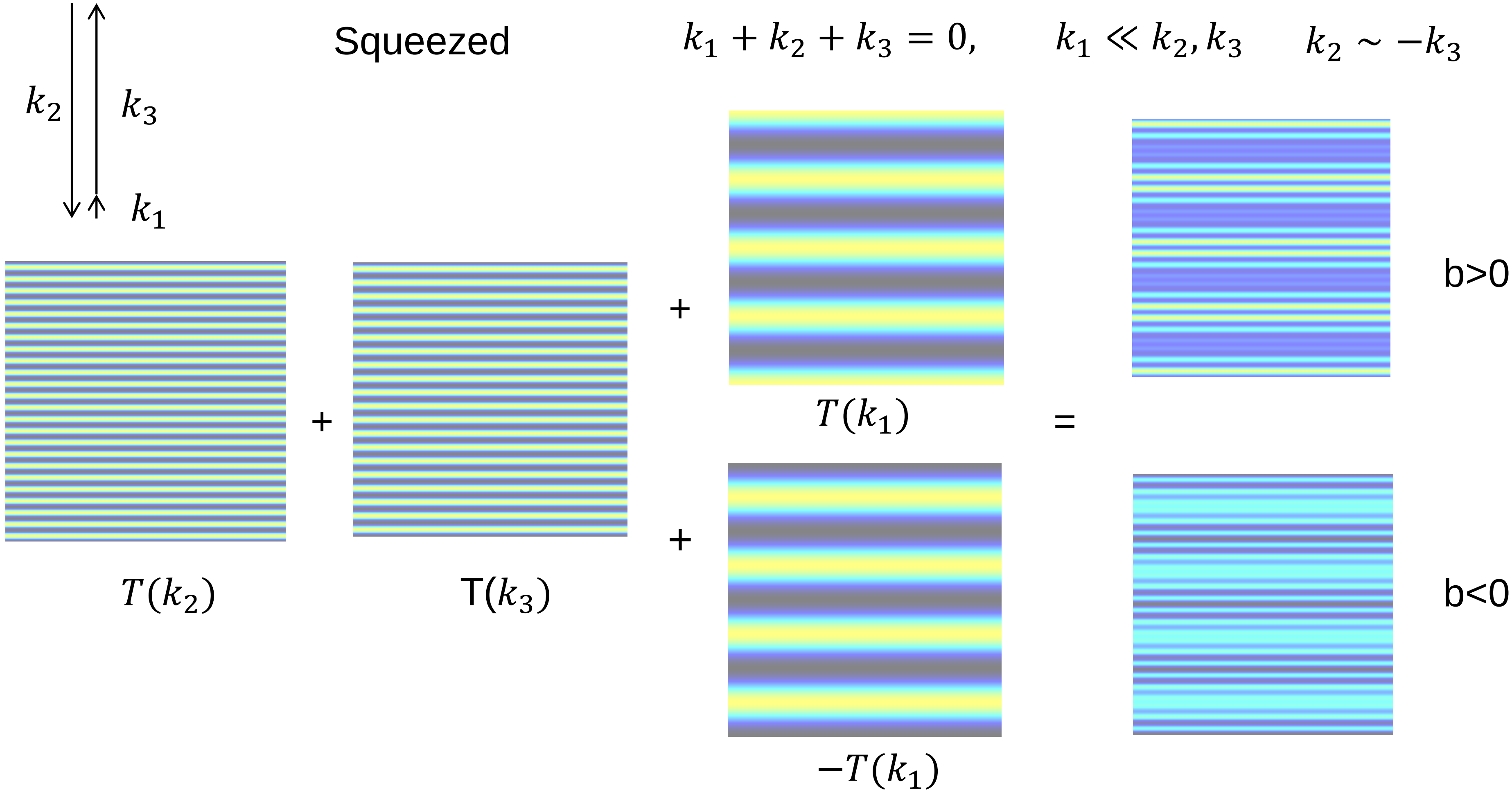
Millennium simulation

Near-equilateral to flattened:









Squeezed bispectrum is a *correlation* of small-scale power with large-scale modes

Calculating a squeezed bispectrum

'Linear-short leg' approximation very accurate for large scales where cosmic variance is large

$l_1 \ll l_2 \le l_3$ with any modulation field(s) X_i so that $\tilde{T} = f(T, X_i)$ where X_i, T gaussian

$$\langle \tilde{T}_{l_1 m_1} \tilde{T}_{l_2 m_2} \tilde{T}_{l_3 m_3} \rangle \approx C_{l_1}^{T X_i} \left\langle \frac{\delta}{\delta X^*_{i, l_1 m_1}} \left(\tilde{T}_{l_2 m_2} \tilde{T}_{l_3 m_3} \right) \right\rangle$$

Correlation of the modulation with the large-scale field

Response of the small-scale power to changes in the modulation field (non perturbative)

Note: uses *only* the linear short leg approximation, otherwise non-perturbatively exact

Example: local primordial non-Gaussianity

Primordial curvature perturbation is modulated as $\zeta = \zeta_0 \left(1 + \frac{6}{5} f_{NL} \zeta_{0,l} \right)$

$l_1 \ll 100$: modulation $\zeta_{0,l}$ super-horizon and constant through last-scattering, $\zeta_0 = \zeta_0^*$

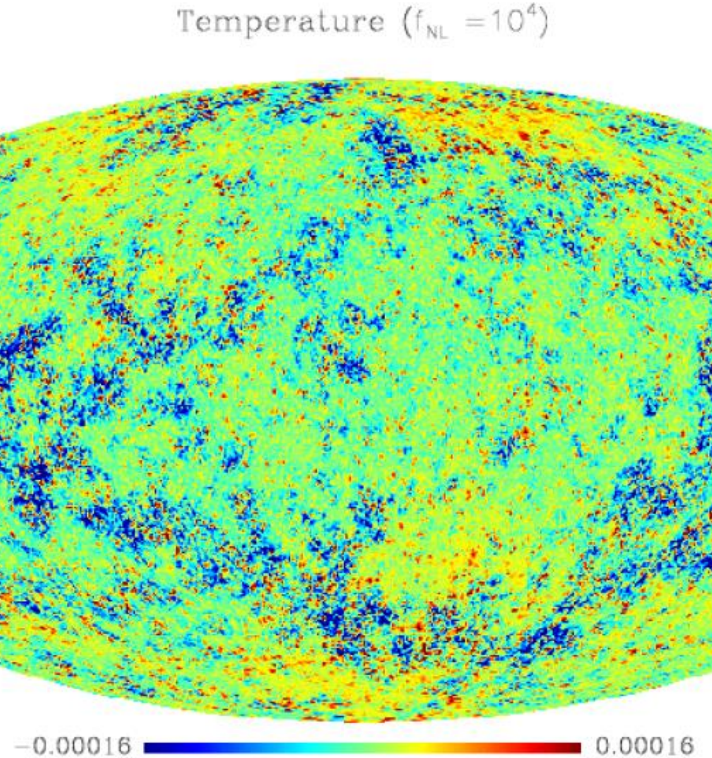
$$\Rightarrow \quad b_{l_1 l_2 l_3} \approx \frac{6}{5} f_{\rm NL} C_{l_1}^{T \zeta_0^*} (\tilde{C}_{l_2} + \tilde{C}_{l_3})$$

(analytic result for $l_1 \ll 100$)

Primordial local non-Gaussianity

e.g. $\zeta = \zeta_0 \left(1 + \frac{6}{5} f_{NL} \zeta_{0,l}\right)$

$$\Rightarrow T \sim T_g (1 + \frac{6}{5} f_{NL} \zeta_{*,l})$$



Liguori et al 2007

Lensing bispectrum calculation

Assume Gaussian fields. Non-perturbative result: $l_1 \leq l_2 \leq l_3$

$$\langle T(\mathbf{l}_1)\tilde{T}(\mathbf{l}_2)\tilde{T}(\mathbf{l}_3)\rangle = C_{l_1}^{T\psi}\left\langle \frac{\delta}{\delta\psi(\mathbf{l}_1)^*}\left(\tilde{T}(\mathbf{l}_2)\tilde{T}(\mathbf{l}_3)\right)\right\rangle$$

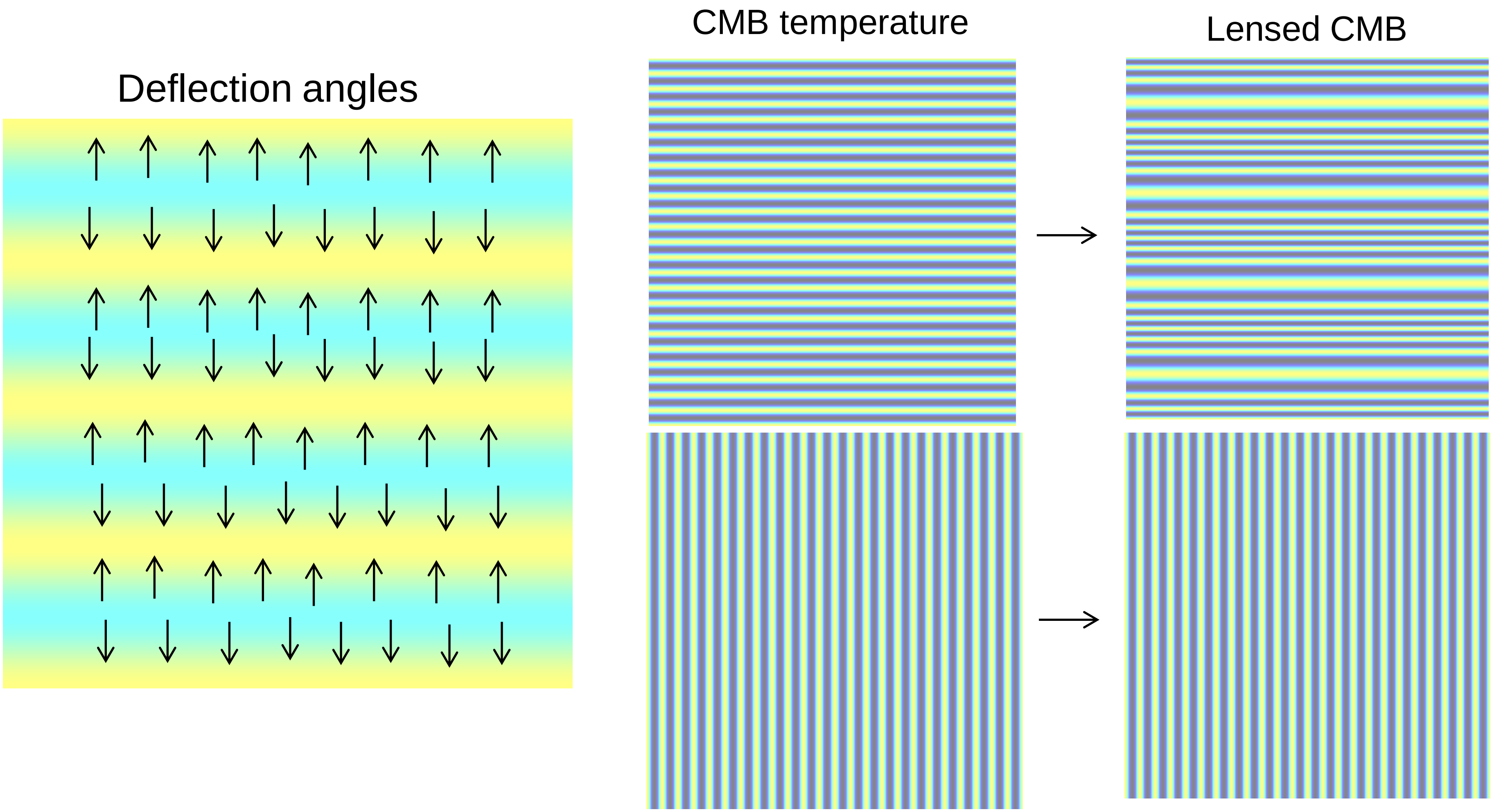
Use $\tilde{T}(\boldsymbol{x}) = T(\boldsymbol{x} + \boldsymbol{\nabla}\psi)$ $\Rightarrow$ $$\frac{\delta}{\delta\psi(\mathbf{l}_1)^*}\tilde{T}(\mathbf{l}) = -\frac{i}{2\pi}\mathbf{l}_1 \cdot \widetilde{\boldsymbol{\nabla} T}(\mathbf{l} + \mathbf{l}_1),$$

$$\begin{aligned}\langle T(\mathbf{l}_1)\tilde{T}(\mathbf{l}_2)\tilde{T}(\mathbf{l}_3)\rangle &= -\frac{i}{2\pi}C_{l_1}^{T\psi}\,\mathbf{l}_1 \cdot \left\langle \widetilde{\boldsymbol{\nabla} T}(\mathbf{l}_1 + \mathbf{l}_2)\tilde{T}(\mathbf{l}_3)\right\rangle + (\mathbf{l}_2 \leftrightarrow \mathbf{l}_3) \\ &= -\frac{1}{2\pi}\delta(\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3)C_{l_1}^{T\psi}\left[(\mathbf{l}_1 \cdot \mathbf{l}_2)\tilde{C}_{l_2}^{T\nabla T} + (\mathbf{l}_1 \cdot \mathbf{l}_3)\tilde{C}_{l_3}^{T\nabla T}\right]\end{aligned}$$

~ Lensed temperature power spectrum

Measuring bispectrum ≡ measuring $C_l^{T\psi}$

Note lensing bispectrum is anisotropic:
small scales modulated in a way that depends on alignment with the large-scale modulating lens

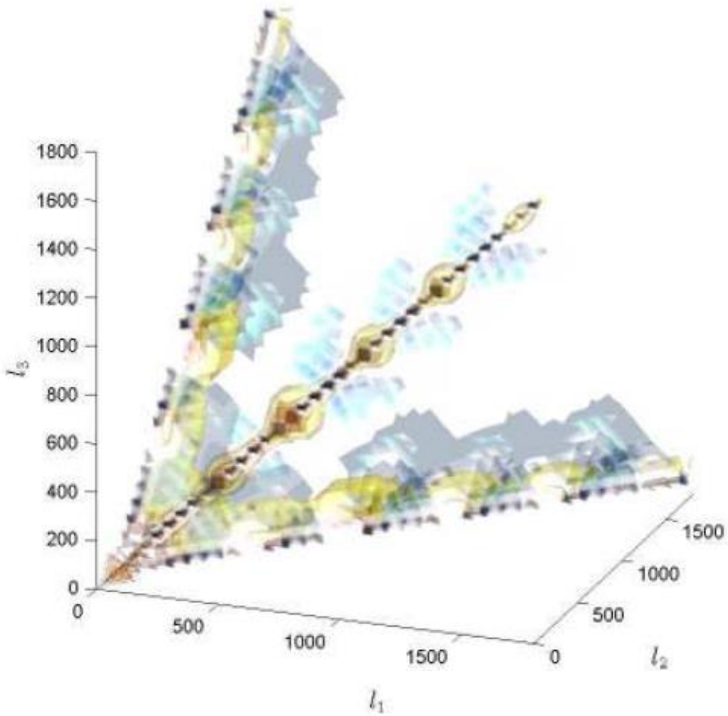


Modulation depends on relative orientation $\Rightarrow$ anisotropic ψTT bispectrum

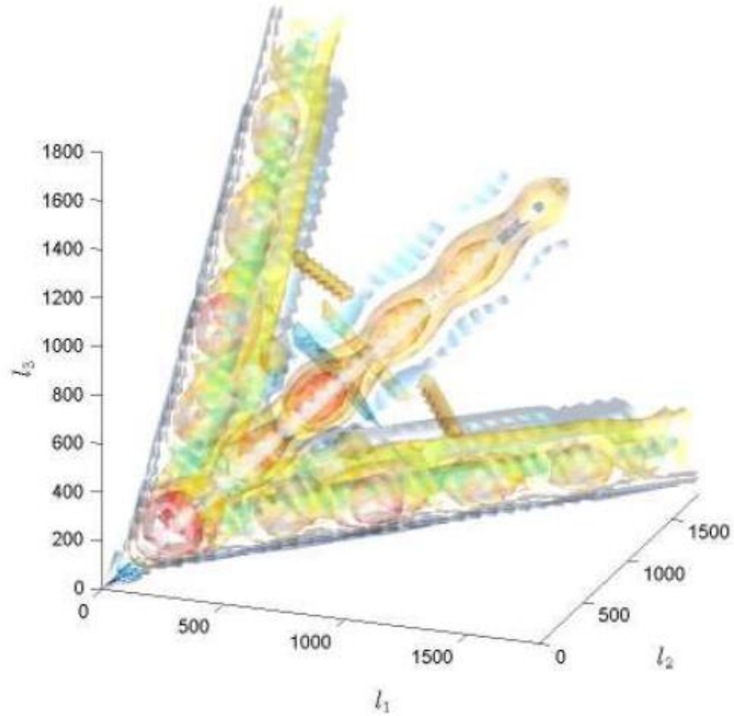
ISW-ψ correlation $\Rightarrow$ $C_l^{T\psi} \neq 0 \Rightarrow$ anisotropic lensing TTT bispectrum

Local modulations (e.g. f_{NL}) are also squeezed, but they are isotropic: no orientation dependence

Squeezed shapes but different phase, angle and scale dependence



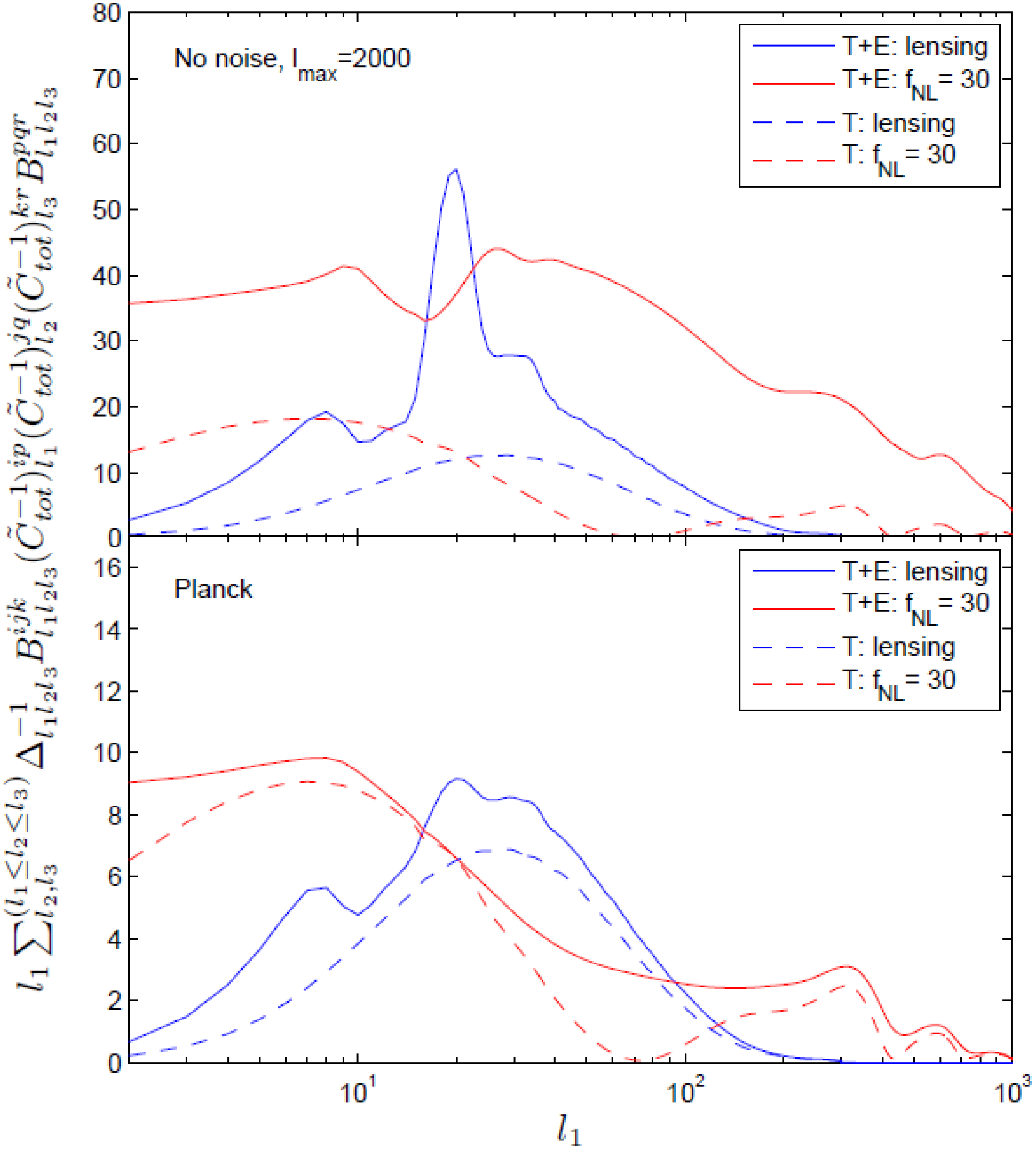
Lensing



f_{NL}

Signal to noise

Contributions to Fisher inverse variance for $b_{l_1 l_2 l_3} = 0$

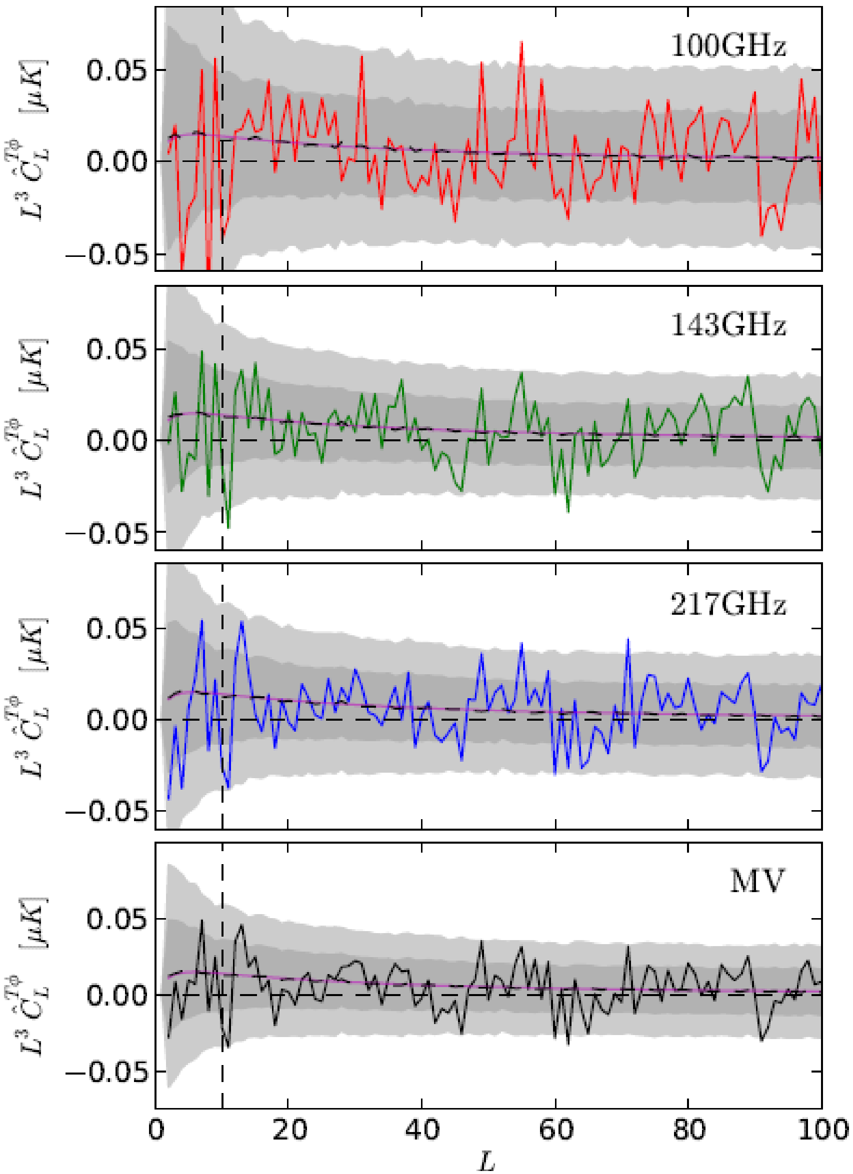


Note: expected $C_l^{T\psi}$ is non-zero: cosmic signal variance

$$\mathrm{Var}(C_l^{T\psi}) = \frac{C_l^{TT} C_l^{\psi\psi} + \left(C_l^{T\psi}\right)^2}{2l+1}$$

In addition to reconstruction noise (standard Fisher result)

Planck lensing bispectrum result



Large cosmic variance and reconstruction noise, but 'detected' at $\sim 2.5\sigma$

Table 2. Results for the amplitude of the ISW-lensing bispectrum from the SMICA, NILC, SEVEM, and C-R foreground-cleaned maps, for the KSW, binned, and modal (polynomial) estimators; error bars are 68% CL.

	SMICA	NILC	SEVEM	C-R
KSW	0.81 ± 0.31	0.85 ± 0.32	0.68 ± 0.32	0.75 ± 0.32
Binned	0.91 ± 0.37	1.03 ± 0.37	0.83 ± 0.39	0.80 ± 0.40
Modal	0.77 ± 0.37	0.93 ± 0.37	0.60 ± 0.37	0.68 ± 0.39

Lensing signal must be subtracted when looking at local f_{NL}

Table 8. Results for the $f_{\rm NL}$ parameters of the primordial local, equilateral, and orthogonal shapes, determined by the KSW estimator from the SMICA foreground-cleaned map. Both independent single-shape results and results marginalized over the point source bispectrum and with the ISW-lensing bias subtracted are reported; error bars are 68% CL.

	Independent KSW	**ISW-lensing subtracted** KSW
SMICA		
Local	9.8 ± 5.8	**2.7 ± 5.8**
Equilateral	-37 ± 75	**-42 ± 75**
Orthogonal	-46 ± 39	**-25 ± 39**

Lensing Trispectrum: Connected four-point **< T T T T>$_c$**

$$\langle\tilde\Theta(\mathbf{l}_1)\tilde\Theta(\mathbf{l}_2)\tilde\Theta(\mathbf{l}_3)\tilde\Theta(\mathbf{l}_4)\rangle_c \equiv \langle\tilde\Theta(\mathbf{l}_1)\tilde\Theta(\mathbf{l}_2)\tilde\Theta(\mathbf{l}_3)\tilde\Theta(\mathbf{l}_4)\rangle - \left[\tilde C^{\Theta}_{l_1}\tilde C^{\Theta}_{l_3}\delta(\mathbf{l}_1+\mathbf{l}_2)\delta(\mathbf{l}_3+\mathbf{l}_4) + \mathrm{perms}\right]$$
$$\approx \frac{1}{2(2\pi)^2}\delta(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3+\mathbf{l}_4)\left[C^{\psi}_{|\mathbf{l}_1+\mathbf{l}_3|}C^{\Theta}_{l_3}C^{\Theta}_{l_4}(\mathbf{l}_1+\mathbf{l}_3)\cdot\mathbf{l}_3\,(\mathbf{l}_2+\mathbf{l}_4)\cdot\mathbf{l}_4 + \mathrm{perms}\right],$$

Bispectrum: measures $C_L^{T\psi}$
Trispectrum: measures $C_L^{\psi\psi}$

Both anisotropic and distinct from primordial signals

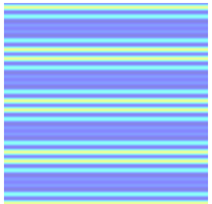
Quadratic estimator TT measures ψ
→ so TTT measures $C_L^{T\psi}$, TTTT measures $C_L^{\psi\psi}$

In exactly the same way, for local primordial modulations $\zeta = \zeta_0(1+\phi)$

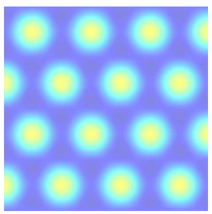
Squeezed f_{NL} Bispectrum: measures $C_L^{T\phi}$ → Only approximate (signal at $L < 500$)
- need to worry about finite last scattering thickness

Squeezed τ_{NL} Trispectrum: measures $C_L^{\phi\phi}$ → Rather accurate (signal L<10)
- Planck τ_{NL} analysis exactly analogous to lensing

Hunters' guide to the non-Gaussianity zoo



Bispectrum



Squeezed

Isotropic (f_{NL}) Local modulations $\chi = \chi_0(1+\phi)$, $P_{\chi_0\phi} \neq 0$

CMB lensing magnification

Anisotropic

CMB lensing

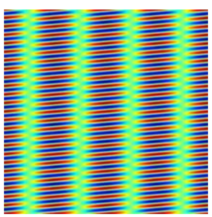
Non-standard anisotropic inflation

Bispectrum = power modulation correlated to temperature $(\chi)(\chi\chi)$

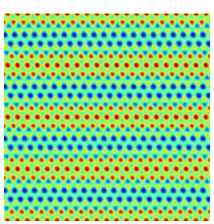
Equilateral-flattened

Dynamically generated
e.g. non-linear structure growth
inflation before horizon exit (e.g. $c_s \ll 1$)

Bispectrum = concentrated hot spots in areas of milder cold (or vice versa)



Trispectrum



Diagonal squeezed

Isotropic(τ_{NL}) Local modulations $\chi = \chi_0(1+\phi)$ (any $P_{\chi_0\phi}$)

CMB lensing magnification

Anisotropic

CMB lensing

Anisotropic primordial power spectrum

Non-standard anisotropic inflation

Gravitational-wave modulation (small)

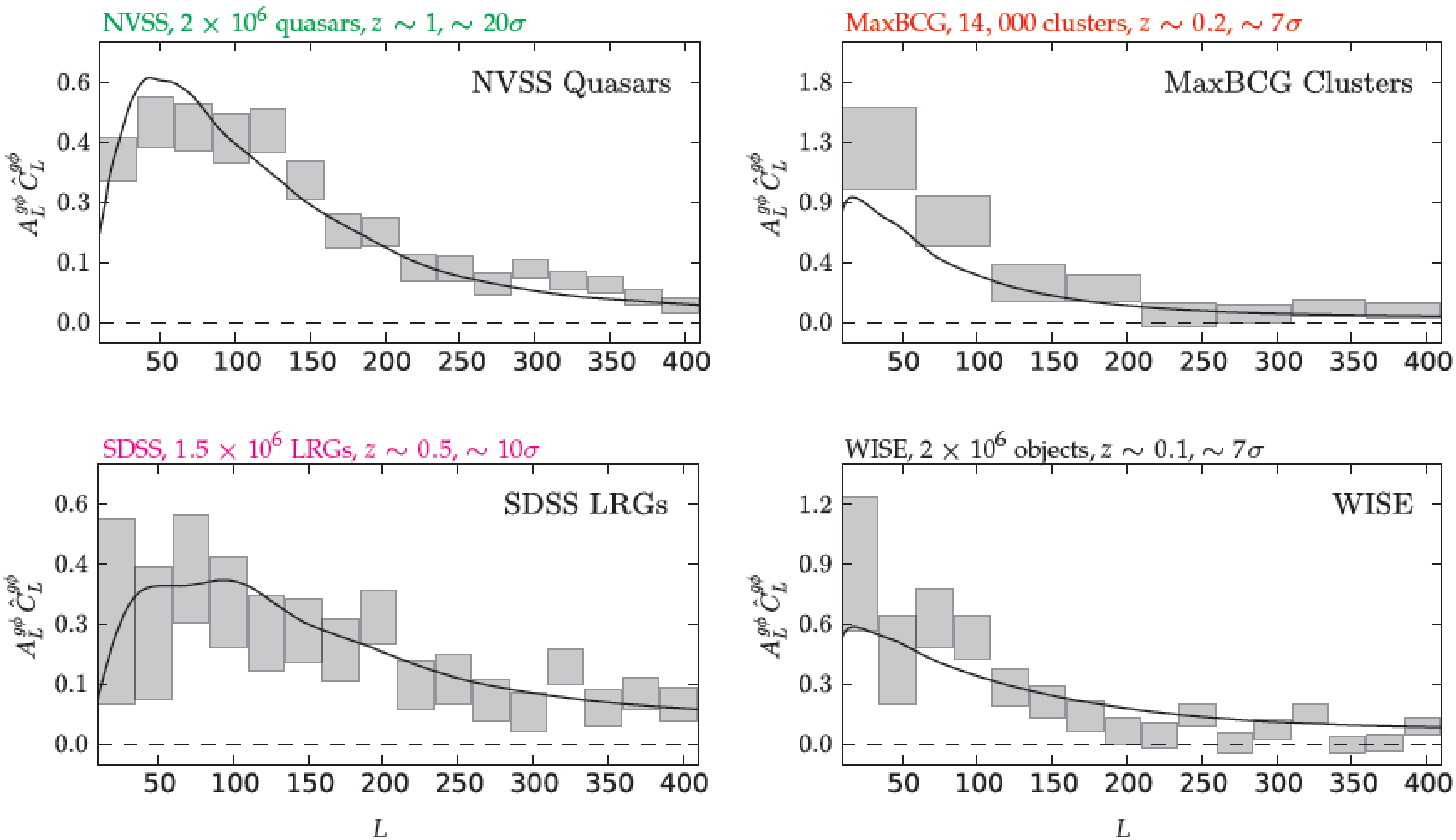
Trispectrum = power spectrum of power modulation, $(\chi\chi)(\chi\chi)$

One-leg squeezed (small: g_{NL} either sign, or τ_{NL} positive)

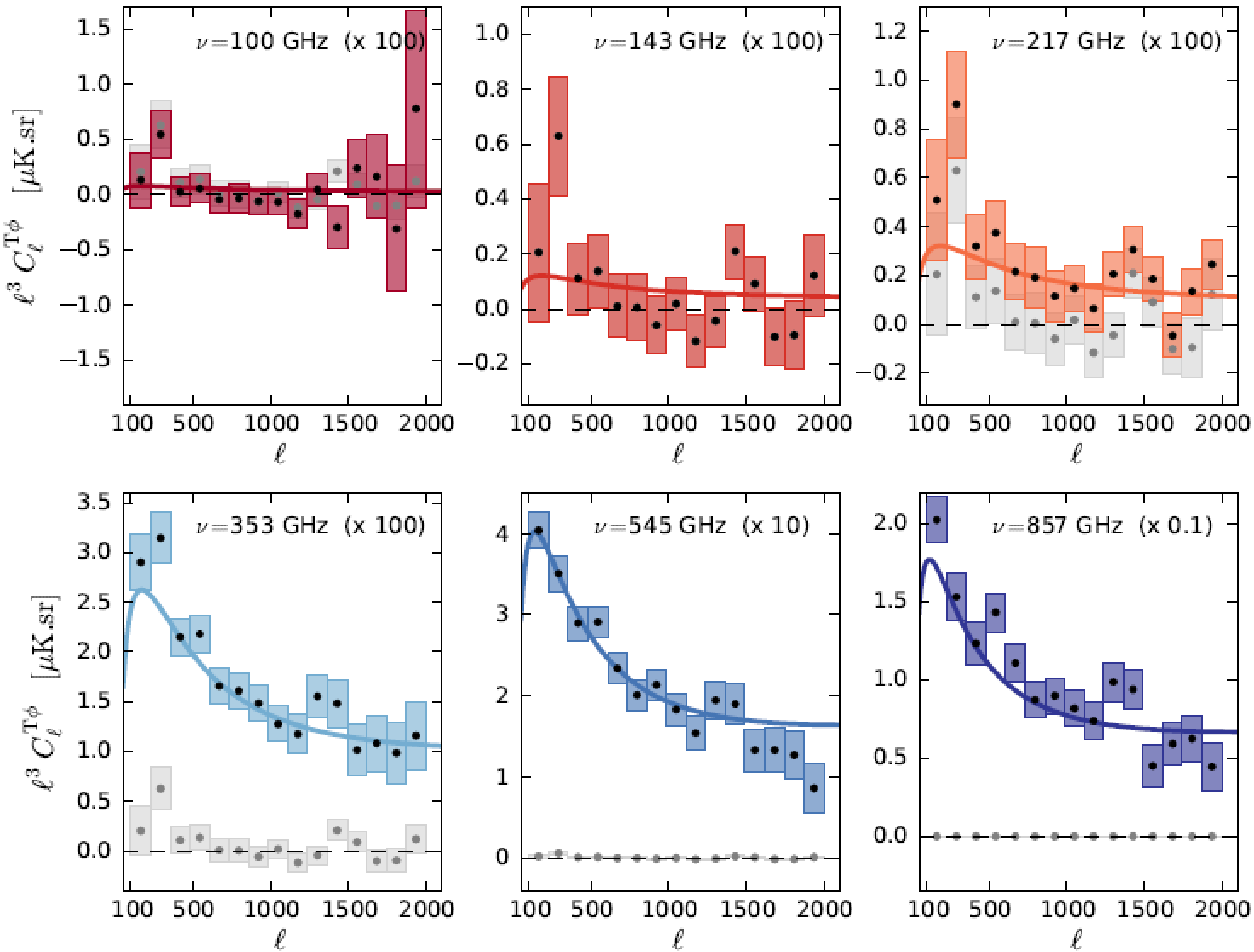
Trispectrum = bispectrum modulation correlated to temperature $(\chi)(\chi\chi\chi)$

…

External Correlations



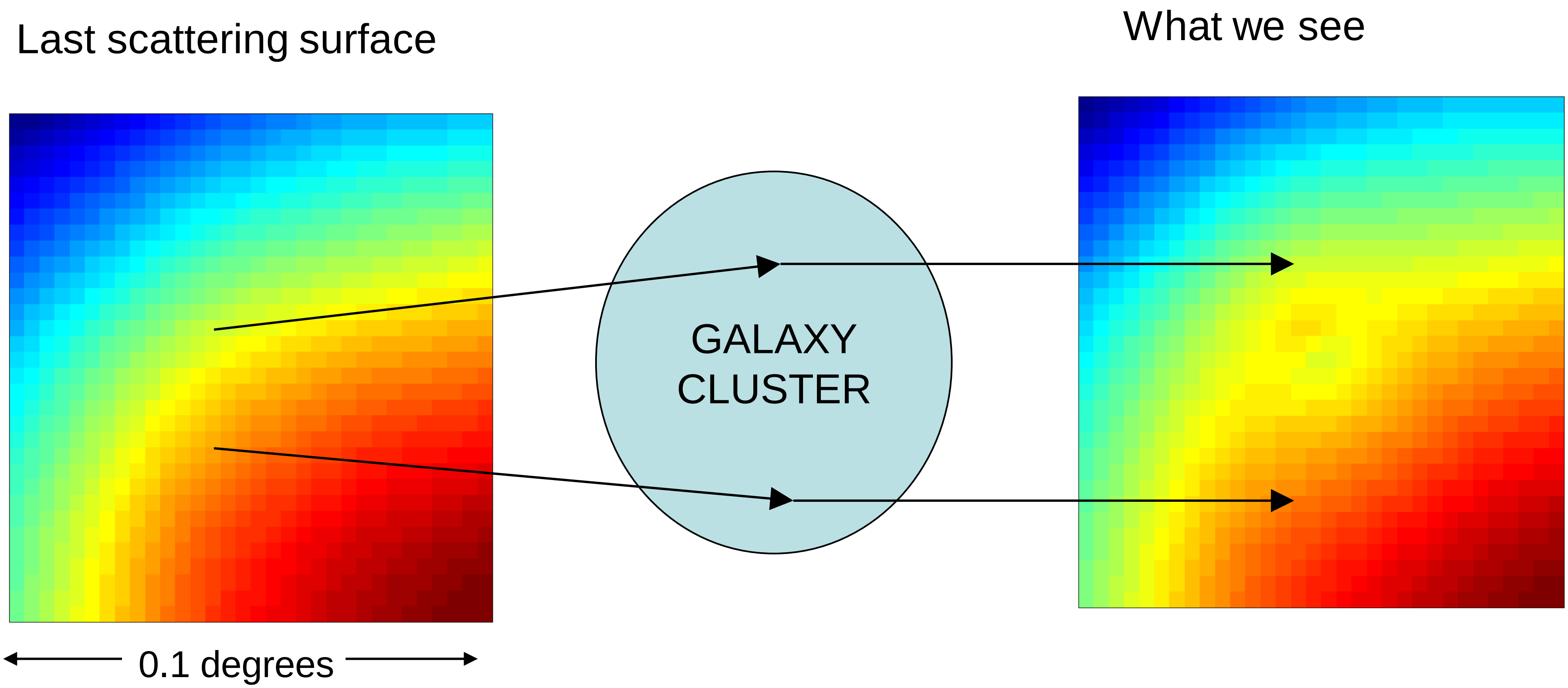
CIB-CMB lensing bispectrum detection



' $42\,\sigma$ at 545 GHz !

Cluster CMB lensing

CMB very smooth on small scales: approximately a gradient



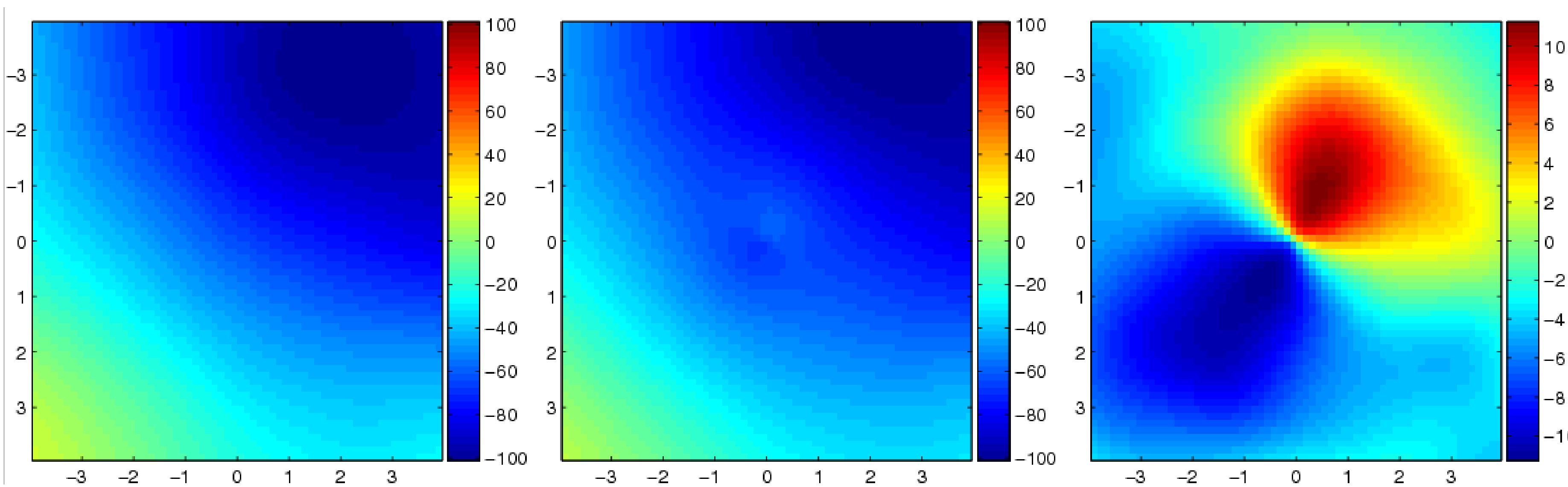
Need sensitive ~ arcminute resolution observations

RMS gradient ~ 13 μK / arcmin
deflection from cluster ~ 1 arcmin

Lensing signal ~ 10 μK

BUT: depends on CMB gradient behind a given cluster

Unlensed Lensed Difference

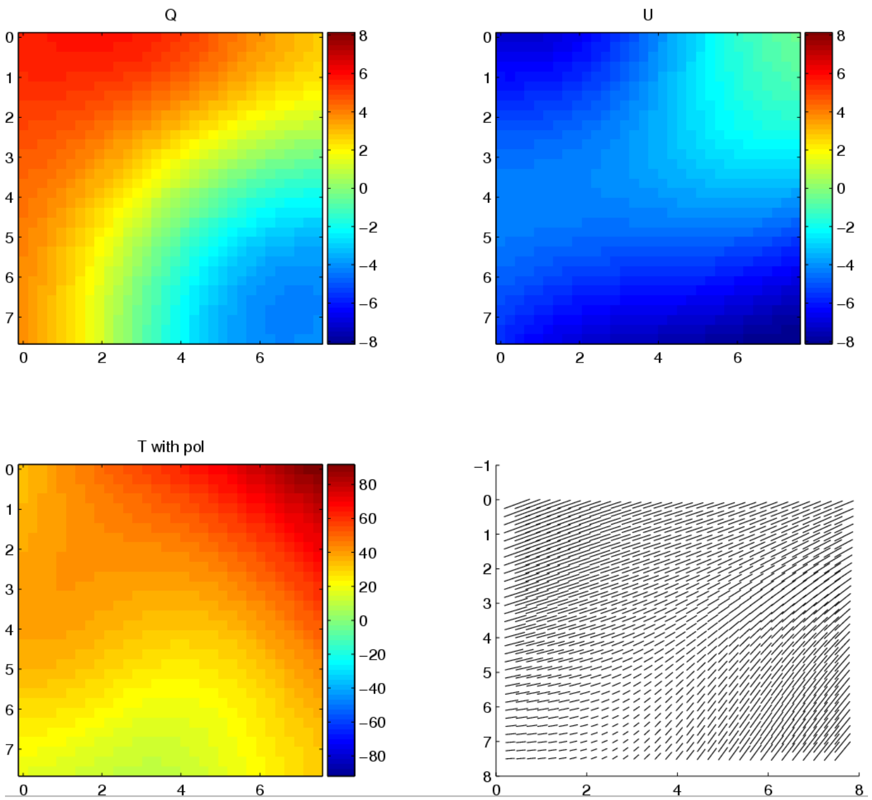


Unlensed CMB unknown, but statistics well understood (background CMB Gaussian) :

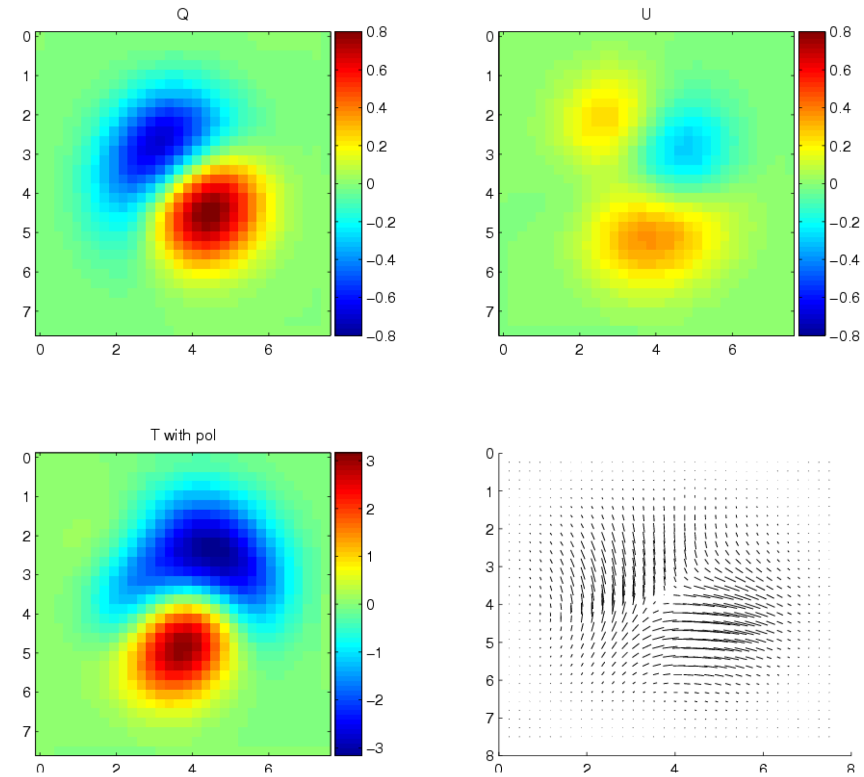
can compute likelihood of given lens (e.g. NFW parameters) essentially exactly

Add polarization observations?

Unlensed T+Q+U



Difference after cluster lensing



Less sample variance – but signal ~10x smaller: need 10x lower noise

Note: E and B equally useful on these scales; gradient could be either

Complications

- Temperature

  - Thermal SZ, dust, etc. (frequency subtractable)
  - Kinetic SZ (big problem?)
  - Moving lens effect (velocity Rees-Sciama, dipole-like)
  - Background Doppler signals
  - Other lenses

- Polarization

  - Quadrupole scattering (< 0.1μK)
  - Re-scattered thermal SZ (freq)
  - Kinetic SZ (higher order)
  - Other lenses

  Generally much cleaner

But usually galaxy lensing does much better, esp. for low redshift clusters