CMB beyond the power spectrum

Antony Lewis
http://cosmologist.info/

Lewis arXiv:1204.5018
Lewis arXiv:1107.5431
Lewis, Challinor & Hanson arXiv:1101.2234
Pearson, Lewis & Regan arXiv:1201.1010
Planck Collaboration 2013
CMB temperature

End of inflation

Last scattering surface

Perturbations super-horizon

Sub-horizon acoustic oscillations
+ modes that are still super-horizon
Observed CMB temperature power spectrum

Primordial perturbations + known physics with unknown parameters

Observations

Constrain theory of early universe
+ evolution parameters and geometry
Beyond Gaussianity – general possibilities

Flat sky approximation: \( \Theta(x) = \frac{1}{2\pi} \int d^2 l \Theta(l) e^{ix \cdot l} \)  
\( (\Theta = T) \)

Gaussian + statistical isotropy

\[ \langle \Theta(l_1)\Theta(l_2) \rangle = \delta(l_1 + l_2)C_l \]

- power spectrum encodes all the information
- modes with different wavenumber are independent

Higher-point correlations

Gaussian: can be written in terms of \( C_l \)

Non-Gaussian: non-zero connected \( n \)-point functions
Flat sky approximation:  \[ \langle \Theta(l_1)\Theta(l_2)\Theta(l_3) \rangle = \frac{1}{2\pi} \delta(l_1 + l_2 + l_3) b_{l_1l_2l_3} \]

If you know \(\Theta(l_1), \Theta(l_2)\), sign of \(b_{l_1l_2l_3}\) tells you which sign of \(\Theta(l_3)\) is more likely

**Bispectrum**

\[ l_1 + l_2 + l_3 = 0 \]

**Trispectrum**

\[ \langle \Theta(l_1)\Theta(l_2)\Theta(l_3)\Theta(l_4) \rangle_C = (2\pi)^{-2} \delta(l_1 + l_2 + l_3 + l_4) T(l_1, l_2, l_3, l_4) \]

\[ \langle \Theta(l_1)\Theta(l_2)\Theta(l_3)\Theta(l_4) \rangle_C = \frac{1}{2} \int \frac{d^2L}{(2\pi)^2} \delta(l_1 + l_2 + L) \delta(l_3 + l_4 - L) T^{(\ell_1\ell_2)}_{(\ell_3\ell_4)}(L) + \text{perms.} \]

**N-spectra**
Equilateral \[ k_1 + k_2 + k_3 = 0, \ |k_1| = |k_2| = |k_3| \]
Millennium simulation
Near-equilateral to flattened:

\[ b > 0 \]

\[ b < 0 \]
Squeezed bispectrum is a *correlation* of small-scale power with large-scale modes.

For more pretty pictures and trispectrum see: [The Real Shape of Non-Gaussianities, arXiv:1107.5431](https://arxiv.org/abs/1107.5431)
Calculating the squeezed bispectrum

‘Linear-short leg’ approximation very accurate for large scales where cosmic variance is large

\[ l_1 \ll l_2 \leq l_3 \quad \text{with modulation field(s) } X_i \]

\[
\left\langle \hat{T}_{l_1 m_1} \hat{T}_{l_2 m_2} \hat{T}_{l_3 m_3} \right\rangle \approx C_{l_1}^{T X_i} \left\langle \frac{\delta}{\delta X_i^*_{l_1 m_1}} \left( \hat{T}_{l_2 m_2} \hat{T}_{l_3 m_3} \right) \right\rangle
\]

Correlation of the modulation with the large-scale field

Response of the small-scale power to changes in the modulation field (non perturbative)

Note: uses only the linear short leg approximation, otherwise non-perturbatively exact

Example: local primordial non-Gaussianity

Primordial curvature perturbation is modulated as

\[ \zeta = [1 + (3f_{NL}/5)\zeta_g]\zeta_g \]

\[ l_1 \ll 100: \text{modulation super-horizon and constant through last-scattering, } \zeta_g = \zeta_0^* \]

\[ b_{l_1 l_2 l_3} \approx \frac{6}{5} f_{NL} C_{l_1}^{T \zeta_0} (\tilde{C}_{l_2} + \tilde{C}_{l_3}) \]
But even with $f_{NL} = 0$, we observe CMB at last scattering modulated by other perturbations
What is the modulating effect of large-scale super-horizon perturbations?

Single-field inflation: only one degree of freedom, e.g. everything determined by local temperature (density) on super-horizon scales

Cannot locally observe super-horizon perturbations (to $O\left(\frac{k^2}{H^2}\right)$)

Observers in different places on LSS will see statistically exactly the same thing (at given fixed temperature/time from hot big bang)
- local physics is identical in Hubble patches that differ only by super-horizon modes
BUT: a distant observer will see modulations due to the large modes \( \sim \) horizon size today - can see and compare multiple different Hubble patches

- Super-horizon modes induce linear perturbations on all scales
  
  linear CMB anisotropies on large scales \((l < 100)\)

- Sub-horizon perturbations are observed in perturbed universe: 
  - small-scale perturbations are modulated by the effect large-scale modes

  squeezed-shape non-Gaussianities
Linear CMB anisotropies

Linear perturbation theory with
\[ ds^2 = a(\eta)^2 \left[ (1 + 2\Psi) d\eta^2 - (1 - 2\Phi) dx^2 \right] \]

Using the geodesic equation in the Conformal Newtonian Gauge:

\[ E(\eta_0) = a(\eta) E(\eta) \left[ 1 + \Psi(\eta) - \Psi_0 + \int_{\eta}^{\eta_0} d\eta (\Psi' + \Phi') \right] \]

All photons redshift the same way, so \( kT \sim E \).

Recombination fairly sharp at background time \( \eta_*: \sim \text{constant temperature} \) surface. Also add Doppler effect:

\[ T(\mathbf{n}, \eta_0) = (a_* + \delta a) T_* \left[ 1 + \Psi(\eta_*) - \Psi_0 + \mathbf{n} \cdot (\mathbf{v}_o - \mathbf{v}) + \int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi') \right] \]

\[ = T_0 \left[ 1 + \frac{\delta a}{a_*} + \Psi(\eta_*) - \Psi_0 + \mathbf{n} \cdot (\mathbf{v}_o - \mathbf{v}) + \int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi') \right] \]
\[ \rho_\gamma \propto T^4 \propto a^4. \]

\[ \Rightarrow \quad \frac{\Delta T_0}{T}(\hat{n}) = \frac{\Delta_\gamma(\eta_*)}{4} + \Psi(\eta_*) - \Psi_0 + \hat{n} \cdot (v_o - v) + \int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi') \]

- Temperature perturbation at recombination
- Sachs-Wolfe
- Doppler
- ISW
Note: no scale on which Sachs-Wolfe $\Phi/3$ result is accurate. Doppler dominates at $l \sim 60$ because other terms cancel.
Alternative

\[ d\Phi = \frac{\partial \Phi}{\partial \eta} \, d\eta + \frac{\partial \Phi}{\partial \chi} \, d\chi \]

\[ a_A E_A = a(\eta)E(\eta) \left[ 1 + \Psi(\eta) - \Psi_A + \int_{\eta}^{\eta_A} \, d\eta \partial_\eta (\Psi + \Phi) \right] \]

\[ = a(\eta)E(\eta) \left[ 1 - \Phi(\eta) + \Phi_A + \int_{\eta}^{\eta_A} \, d\eta \partial_\chi (\Psi + \Phi) \right] \]

\[ \Delta T(\hat{n}) = \frac{\Delta \gamma}{4} - \Phi + \hat{n} \cdot (v_A - v) + \int_{\eta}^{\eta_A} \, d\eta \hat{n} \cdot \nabla (\Psi + \Phi) \]

\[ = \zeta_\gamma + \hat{n} \cdot (v_A - v) + \int_{\eta}^{\eta_A} \, d\eta \hat{n} \cdot \nabla (\Psi + \Phi) \]

Gauge-invariant 3-curvature on constant temperature hypersurfaces;
Redshifting from $\delta N$ expansion of the beam makes the $\delta N$ expansion from inflation observable
(but line of sight integral is larger on large-scales: overdensity looks colder)
Non-linear effect due to redshifting by large-scale modes?

Large-scale linear anisotropies are due to the linear anisotropic redshifting of the otherwise uniform (zero-order) temperature last scattering surface

\[ T \to (1 + \Delta T)T \]

Also non-linear effect due to the linear anisotropic redshifting of the linear last scattering surface

\[ \Delta T_{\text{small}} \to (1 + \Delta T_{\text{large}})\Delta T_{\text{small}} \]

Reduced bispectrum

\[ b_{l_1l_2l_3} \approx C_{l_1} \left( \tilde{C}_{l_2} + \tilde{C}_{l_3} \right) \]

Large-scale power spectrum

Small-scale (non-perturbative) power spectrum

(Actually very small, so not very important)
Linear effects of large-scale modes

- Redshifting as photons travel through perturbed universe

- Transverse directions also affected:
  perturbations at last scattering are distorted as well as anisotropically redshifted
Jabobi map relates observed angle to physical separation of pair of rays

\[ \xi_I(\lambda) = D_{IJ}(\lambda) \delta \theta_J \]

Physical separation vector orthogonal to ray

Jacobi map

Angular separation seen by observer at \( A \)

Evolution of Jacobi map:

\[ \frac{d^2 D_{IJ}}{d\lambda^2} = T_{IK}D_{KJ} \]

Optical tidal matrix depends on the Riemann tensor:

\[ T_{IJ} \equiv -E^b_I E^c_J k^a k^d R_{abcd} \]

\( (k^a \) is wave vector along ray, \( E^a_I \) projects into ray-orthogonal basis)
‘Riemann = Weyl + Ricci’

Non-local part (does not depend on local density):
- e.g. determined by Weyl (Newtonian) potential $\frac{1}{2}(\Phi + \Psi)$

- differential deflection of light rays ⇒ convergence and shear of beam

Einstein equations relate Ricci to stress-energy tensor: depends on local density

⇒ ray area changes due to expansion of spacetime as the light propagates

- Ricci focussing

(can be modelled as transverse deflection angle)

FRW background universe has Weyl=0, Ricci gives standard angular diameter distance

At radial distance $\chi_*$, trace of Jacobi map determines physical areas: $D/2 = \chi_* a_*$
Beam propagation in a perturbed universe, e.g. Conformal Newtonian Gauge

\[ ds^2 = a^2(\eta)[(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^idx^j] \]

\[ \xi_I(\lambda) = D_{I,J}(\lambda)\delta\theta_J \]

Trace-free part of Jacobi map depends on the shear:

\[ D_{(IJ)} = a_*\chi_*\gamma_{IJ} \]

\[ \gamma_{IJ} = \nabla_{(I}\nabla_{J)}\psi \quad \psi \equiv -2 \int_0^{\chi_*} d\chi \frac{\chi_* - \chi}{\chi\chi_*} \Psi_W(\chi\hat{n}, \eta_A - \chi) \]

Area of beam determined by trace of Jacobi map:

\[ D(\hat{n}, \eta)/2 = \chi(\hat{n}, \eta)a(\eta)[1 + \Phi_A - \Phi - \kappa + \hat{n} \cdot v_A] \]

CMB is constant temperature surface:

\[ \eta = \eta_* + \delta\eta \]

\[ \rho_{\gamma} \propto T^4 \propto a^4 \]

Radial displacement (small, \( \delta\chi \ll \chi_* \))

\[ \zeta_\gamma \equiv \Delta\gamma/4 - \Phi \]

Ricci focussing

(Weyl) convergence

Local aberration
\[ \frac{D}{2} \approx \chi_* a_*(1 + \zeta_\gamma - \kappa) \]

Overdensity ($\zeta$ larger)

Ricci focussing: beam contracts more leaving LSS $\Rightarrow$ same physical size looks smaller

(Weyl lensing effect not shown and partly cancels area effect)
Gauge-invariant Ricci focussing \( \zeta_\gamma \equiv \Delta_\gamma/4 - \Phi \)
Observable CMB bispectrum from single-field inflation

Linear-short leg approximation for nearly-squeezed shapes:

\[
\langle \tilde{T}_{l_1 m_1} \tilde{T}_{l_2 m_2} \tilde{T}_{l_3 m_3} \rangle \approx C_{l_1}^{TX_i} \left( \frac{\delta}{\delta X_{i, l_1 m_1}} (\tilde{T}_{l_2 m_2} \tilde{T}_{l_3 m_3}) \right)
\]

Where \(X_i\) here is \(\delta T, \kappa\) and \(\zeta_\gamma\), with \(\frac{D}{2} \approx \chi_* a_* (1 + \zeta_\gamma - \kappa)\). For super-horizon adiabatic modes \(\zeta_\gamma = \zeta_0\).

Weyl lensing bispectrum

\[
b_{l_1 l_2 l_3} = \frac{1}{2} \left[ (l_1(l_1 + 1) + l_2(l_2 + 1) - l_3(l_3 + 1)) C_{l_1}^{T_T} \tilde{C}_{l_2} \right] + \text{perms}
\]

Squeezed limit \((l_1 \ll l)\)

\[
b_{l_1 l_2 l_3} \approx C_{l_1}^{T_T} \left[ \frac{1}{l^2} \frac{d(l^2 \tilde{C}_l)}{d \ln l} + \cos 2\phi_{l_1 l} \frac{d \tilde{C}_l}{d \ln l} \right]
\]

Ricci focussing bispectrum

\[
b_{l_1 l_2 l_3} \approx C_{l_1}^{T_T} \frac{1}{2} \frac{d}{d \ln X_*} \left[ \tilde{C}_{l_2} + \tilde{C}_{l_3} \right]
\]

Squeezed limit \((l_1 \ll l)\)

\[
b_{l_1 l_2 l_3} \approx -C_{l_1}^{T_T} \frac{1}{l^2} \frac{d}{d \ln l} (l^2 \tilde{C}_l)
\]

+ anisotropic redshifting bispectrum (from before)

\[
b_{l_1 l_2 l_3} \approx C_{l_1} \left( \tilde{C}_{l_2} + \tilde{C}_{l_3} \right)
\]
Weyl lensing depends on cross-correlation, $C_l^{T\psi}$

(note Rees-Sciama contribution is small, numerical problem with much larger result of Verde et al, Mangilli et al.; see also Junk et al. 2012 who agree with me)
Does this look like squeezed non-Gaussianity $f_{NL}$ from multi-field inflation (local modulation of small scale perturbation amplitudes in each Hubble patch)?

<table>
<thead>
<tr>
<th>Data used</th>
<th>$\sigma_{f_{NL}}$</th>
<th>Weyl</th>
<th>Ricci</th>
<th>Redshift</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>4.3</td>
<td>9.5</td>
<td>1.5</td>
<td>-0.22</td>
<td>10.7</td>
</tr>
<tr>
<td>Planck $T$</td>
<td>5.9</td>
<td>6.4</td>
<td>1.0</td>
<td>-0.22</td>
<td>7.1</td>
</tr>
<tr>
<td>$T \ (l_1 &lt; 60)$</td>
<td>4.6</td>
<td>10.6</td>
<td>1.7</td>
<td>-0.25</td>
<td>12.0</td>
</tr>
<tr>
<td>Planck $T \ (l_1 &lt; 60)$</td>
<td>6.2</td>
<td>7.0</td>
<td>1.1</td>
<td>-0.25</td>
<td>7.9</td>
</tr>
<tr>
<td>$T+E$</td>
<td>2.1</td>
<td>2.6</td>
<td>1.1</td>
<td>-0.05</td>
<td>3.7</td>
</tr>
<tr>
<td>Planck $T+E$</td>
<td>5.2</td>
<td>4.3</td>
<td>1.0</td>
<td>-0.15</td>
<td>5.2</td>
</tr>
</tbody>
</table>

Dominated by lensing $f_{NL} \sim 6 - 10$

Ricci is an $O(1)$ correction

Calculation reliable for $l_1 < 60$ where dynamical effects suppressed by small $\frac{k^2}{H^2}$: do not need fully non-linear dynamical calculation of bispectrum a la Pitrou et al to make reliable $f_{NL}$ constraint.

TABLE I: Individual and total biases on primordial local-model non-Gaussianity parameterized by $f_{NL}$ for CMB temperature and $E$-polarization data with Planck-like noise (assuming isotropic coverage over the full sky with sensitivity $\Delta T = \Delta Q/2 = \Delta U/2 = 50 \mu$K arcmin [$N^T_i = N^E_i / 4 = 2 \times 10^{-4} \mu$K$^2$] and a beam FWHM of 7 arcmin) or cosmic-variance limited data with $l_{\text{max}} = 2000$. Results are assuming that non-$f_{NL}$ contributions are only significant at $l_1 \leq 300$ and negligible dynamical effects; the $l_1 < 60$ results are filtered to only use large scale modulations and are therefore immune to small-scale modulation effects. The bias is the systematic error on $f_{NL}$ if the given contribution is neglected, which can be compared to $\sigma_{f_{NL}}$ which is the Fisher error estimate (including lensing signal variance).
Signal easily modelled
Squeezed shape but different phase, angle and scale dependence

Lensing

$f_{NL}$

Lewis, Challinor, Hanson 1101.2234
Note: ‘Maldacena’ bispectrum

\[ \langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle \approx -\frac{1}{(2\pi)^3/2} \delta(k_1 + k_2 + k_3)P_\zeta(k_1) \frac{1}{2} \left[ \frac{1}{k_3^3} \frac{d}{d\ln k_3} (k_3^3 P_\zeta(k_3)) + \frac{1}{k_2^3} \frac{d}{d\ln k_2} (k_2^3 P_\zeta(k_2)) \right] \]

Consistency relation: \( f_{NL} \sim O(n_s - 1) \)

is not an observable
- cannot measure comoving curvature perturbations on scales larger than the horizon directly
- \( d/d(\ln k) \) and CMB transfer functions do not commute: cannot get correct result from primordial \( f_{NL} \sim (n_s - 1) \)

Observable CMB analogue is Ricci focussing bispectrum

\[ b_{l_1 l_2 l_3} \approx -C_{l_1}^T \zeta^* \frac{1}{l^2} \frac{d}{d\ln l} (l^2 \tilde{C}_l) \]

- larger because of acoustic oscillations, non-zero for \( n_s = 1 \)
- different shape to \( f_{NL} \) in CMB, but projects as \( f_{NL} = O(1) \)
Table 2. Results for the amplitude of the ISW-lensing bispectrum from the SMICA, NILC, SEVEM, and C-R foreground-cleaned maps, for the KSW, binned, and modal (polynomial) estimators; error bars are 68% CL.

<table>
<thead>
<tr>
<th></th>
<th>SMICA</th>
<th>NILC</th>
<th>SEVEM</th>
<th>C-R</th>
</tr>
</thead>
<tbody>
<tr>
<td>KSW</td>
<td>0.81 ± 0.31</td>
<td>0.85 ± 0.32</td>
<td>0.68 ± 0.32</td>
<td>0.75 ± 0.32</td>
</tr>
<tr>
<td>Binned</td>
<td>0.91 ± 0.37</td>
<td>1.03 ± 0.37</td>
<td>0.83 ± 0.39</td>
<td>0.80 ± 0.40</td>
</tr>
<tr>
<td>Modal</td>
<td>0.77 ± 0.37</td>
<td>0.93 ± 0.37</td>
<td>0.60 ± 0.37</td>
<td>0.68 ± 0.39</td>
</tr>
</tbody>
</table>

Table 8. Results for the $f_{\text{NL}}$ parameters of the primordial local, equilateral, and orthogonal shapes, determined by the KSW estimator from the SMICA foreground-cleaned map. Both independent single-shape results and results marginalized over the point source bispectrum and with the ISW-lensing bias subtracted are reported; error bars are 68% CL.

<table>
<thead>
<tr>
<th></th>
<th>Independent KSW</th>
<th>ISW-lensing subtracted KSW</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMICA</td>
<td>SMICA</td>
<td></td>
</tr>
<tr>
<td>Local</td>
<td>9.8 ± 5.8</td>
<td>2.7 ± 5.8</td>
</tr>
<tr>
<td>Equilateral</td>
<td>-37 ± 75</td>
<td>-42 ± 75</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>-46 ± 39</td>
<td>-25 ± 39</td>
</tr>
</tbody>
</table>
Diagonal squeezed trispectra

\[ |k_1| \sim |k_2|, \ |k_3| \sim |k_4|, \ |k_1 + k_2| = |k_3 + k_4| \ll |k_2|, |k_3| \]

Trispectrum = power spectrum of modulation

e.g. \( \chi = \chi_0 (1 + f_{NL} \chi_0) \)

\[ \tau_{NL} \sim f_{NL}^2 \]

or \( \chi = \chi_0 (1 + \phi) \)

(any correlation, \( \tau_{NL} > f_{NL}^2 \))
One-leg squeezed trispectra

\[ |k_1| \ll |k_2| \sim |k_3| \sim |k_4| \]

Correlated modulation of equilateral bispectrum

e.g. from \( \chi = \chi_0 (1 + g_{NL} \chi_0^2) \) [no diagonal squeezed]

also from \( \chi = \chi_0 (1 + f_{NL} \chi_0) \) [plus diagonal squeezed]
Reconstructing the modulation field

Marginalized over modulation field:

\[ T \sim \int P(T, \psi) d\psi \]

- Non-Gaussian statistically isotropic temperature distribution
- Bispectrum + significant connected 4-point function

For a given modulation field:

\[ T \sim P(T|\psi) \]

- Anisotropic Gaussian temperature distribution
- Re-construct the modulation field
Anisotropy estimators – e.g. reconstruct the modulating lensing field
Reconstructing the modulation field

For a given (fixed) modulation field:

\[ T \sim P(T|X) \]

- Anisotropic Gaussian temperature distribution

- Modes correlated for \( k_2 \neq k_3 \)

\[
\langle \hat{T}(k_2)\hat{T}(k_3) \rangle_{P(\hat{T}|X)} \approx \int dK \hat{X}(K)^* \left\langle \frac{\delta}{\delta X(K)^*} \left( \hat{T}(k_2)\hat{T}(k_3) \right) \right\rangle \approx A(K, k_2, k_3) X(K)^*|_{K=-k_2-k_3} \]

\[ A(K, k_2, k_3)\delta(K + k_2 + k_3) \]

Model-dependent function you can calculate for \( X(K) = 0 \)

Can reconstruct the modulation field \( X \! \! \!)!

- For small modulations can construct “optimal” QML estimator \( \hat{X}(K) \) by summing filtered fields appropriately over \( k_2, k_3 \)

\[
\hat{X}(K) \sim N \left( \sum_{k_2, k_3} A(K, k_2, k_3) \bar{T}(k_2)\bar{T}(k_3) - \text{mean field} \right)
\]

Zaldarriaga, Hu, Hanson, etc..
Planck lensing potential reconstruction (north and south galactic)

Note – about half signal, half noise, not all structures are real map is effectively Wiener filtered

Correlation with $T \Rightarrow$ Bispectrum; Power spectrum $\Rightarrow$ Trispectrum
Primordial modulations?

Primordial curvature modulation:
\[ \zeta(x) = \zeta_0(x)[1 + \phi(x)] \]

Squeezed shape \(\Rightarrow\) large-scale modulation

\[ T(\hat{n}) \approx T_g(\hat{n})[1 + \phi(\hat{n}, r_*)] \equiv T_g(\hat{n})[1 + f(\hat{n})] \]
Complication: Kinematic dipole modulation signal

Modulation

\[ \Delta \Theta(\hat{n}) \rightarrow \left[ 1 + \hat{n} \cdot v + T \frac{d^2 I_v}{dT^2} \hat{n} \cdot v \right] \Delta \Theta(\hat{n}) \]
\[ = (1 + [x \coth(x/2) - 1] \hat{n} \cdot v) \Delta \Theta(\hat{n}), \]
\[ x \equiv h \nu / k_b T \]

Aberration

\[ \hat{n} \rightarrow \hat{n} + \nabla (\hat{n} \cdot v) \]
- just like a dipole lensing convergence

Illustrated for \( \frac{\nu}{c} = 0.85 \)

Challinor & van Leeuwen 2002
known dipole amplitude and direction

\[ \frac{v}{c} \approx 1.23 \times 10^{-3} \]

Modulation

\[ f = \left( x \coth \left( \frac{x}{2} \right) - 1 \right) \hat{n} \cdot \nu \equiv b_v \hat{n} \cdot \nu \]

Approx boost factor

\[ b_v = x \coth \left( \frac{x}{2} \right) / 2 - 1 \]

Map modulation amplitude

<table>
<thead>
<tr>
<th>Planck maps:</th>
<th>100 GHz</th>
<th>143 GHz</th>
<th>217 GHz</th>
<th>353 GHz</th>
<th>545 GHz</th>
<th>857 GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.5</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>14</td>
</tr>
<tr>
<td>Approx boost factor</td>
<td>0.18%</td>
<td>0.24%</td>
<td>0.37%</td>
<td>0.64%</td>
<td>1.1%</td>
<td>1.7%</td>
</tr>
</tbody>
</table>

Use 143, 217 only (with dust subtraction from 857)

*Note*: SMICA maps are a complicated mixture; modulation effect not currently included in FFP6 sims
143x217 modulation reconstruction ($L \leq 5$)
- map of estimated modulation field $f$

Kinematics not subtracted

Kinematics subtracted in mean field from sims
Dipole kinematic effect using appropriate quadratic estimators

\[
\hat{\beta}_|| \quad \hat{\beta}_\perp \quad \hat{\beta}_x \\
\hat{\phi}_|| \quad \hat{\phi}_\perp \quad \hat{\phi}_x \\
\hat{\tau}_|| \quad \hat{\tau}_\perp \quad \hat{\tau}_x
\]

Simulations without velocity effects (143 × 217)
Simulations with velocity effects

- 5σ detection in 143 × 217: \( v_\parallel = (384 \pm 78) \text{ km s}^{-1} \)

- Foreground issue at 217 × 217 in \( \hat{\beta}_x \) (driven by \( \hat{\tau}_x \))?

Note: not included in parameter analysis \( \theta_* = (1.04148 \pm 0.00066) \times 10^{-2} = 0.596724^\circ \pm 0.00038^\circ \)

- bias due to aberration average over mask \( \sim 0.25\sigma \)
Modulation pseudo-power spectrum

\[ \tau_{NL}(L) \equiv \frac{C_L^f}{C_L^{f*}} \]

Consistent with zero except for anomalous octopole
Kinematic subtracted

Kinematics not subtracted

Power dipole directions ($l \leq l_{\text{max}}$)

(as in Doppler paper but here pure modulation estimator)
Power modulation dipole? Result for amplitude at $l \leq l_{\text{max}}$

**WMAP 5 (Hanson & Lewis 2009)**

- Modulation < 1% for $l \leq l_{\text{max}} = 500$

**Planck 217x143 (kinematic subtracted)**

- Modulation < 0.2% for $l_{\text{max}} = 1500 - 2000$
Using full resolution to estimate local trispectrum

Local trispectrum often measured by

\[
\hat{\tau}_{NL} \approx N^{-1} \sum_{L=L_{\text{min}}}^{L_{\text{max}}} \frac{2L + 1}{L^2(L + 1)^2} \frac{\hat{C}_L^f}{\hat{C}_L^\zeta^*},
\]

(optimal to percent level)

Conventional normalization to primordial power

**Planck** \( \tau_{NL} \) trispectrum constraint

Estimator result \( \hat{\tau}_{NL} = 442 \).

Gaussian simulations:

\[-452 < \hat{\tau}_{NL} < 835 \text{ at } 95\% \text{ CL} \quad (\sigma_{\tau_{NL}} \approx 335)\]

Consistent with Gaussian null hypothesis (octopole has small weight)

Conservative upper limit, allowing octopole to be physical using Bayesian posterior

\[\tau_{NL} < 2800 \text{ at } 95\% \text{ CL}\]
Conclusions

• Single field inflation predicts significant non-Gaussianity in the observed CMB
  - mostly due to (Weyl) lensing
  - total projects onto $f_{NL} \sim 7$ for Planck temperature
  - Ricci focussing expansion of beam recovers the $\delta N$ from inflation,
    : gives equivalent of consistency relation, but larger
    : small and not quite observable, projects on to $f_{NL} \sim 1$

• Lensing bispectrum signal important but distinctive shape
  - Marginally detected by Planck

• Local Doppler signal predicted in trispectrum (dipole modulation)

• No sign yet of primordial NG once lensing and Doppler signals subtracted..