Primordial squeezed non-Gaussianity and observables in the CMB

Antony Lewis
http://cosmologist.info/

Lewis arXiv:1204.5018
see also Creminelli et al astro-ph/0405428,

Lewis arXiv:1107.5431
Lewis, Challinor & Hanson arXiv:1101.2234
Pearson, Lewis & Regan arXiv:1201.1010

Benasque August 2012
CMB temperature

End of inflation

Perturbations super-horizon

Gravity + pressure + diffusion

Last scattering surface

Sub-horizon acoustic oscillations + modes that are still super-horizon
$z \approx 1000$

$z = 0$

14 000 Mpc

$\theta$
Observed CMB temperature power spectrum

Observations

Constrain theory of early universe + evolution parameters and geometry
Beyond Gaussianity – general possibilities

Flat sky approximation:  \( \theta(x) = \frac{1}{2\pi} \int d^2l \, \theta(l) e^{ix \cdot l} \)  
\( (\theta = T) \)

Gaussian + statistical isotropy

\[ \langle \theta(l_1)\theta(l_2) \rangle = \delta(l_1 + l_2)C_l \]

- power spectrum encodes all the information
- modes with different wavenumber are independent

Higher-point correlations

Gaussian: can be written in terms of \( C_l \)

Non-Gaussian: non-zero connected \( n \)-point functions
Flat sky approximation:  
\[ \langle \Theta(l_1)\Theta(l_2)\Theta(l_3) \rangle = \frac{1}{2\pi} \delta(l_1 + l_2 + l_3) b_{l_1l_2l_3} \]

If you know \( \Theta(l_1), \Theta(l_2) \), sign of \( b_{l_1l_2l_3} \) tells you which sign of \( \Theta(l_3) \) is more likely

Trispectrum

\[ \langle \Theta(l_1)\Theta(l_2)\Theta(l_3)\Theta(l_4) \rangle_C = (2\pi)^{-2} \delta(l_1 + l_2 + l_3 + l_4) T(l_1, l_2, l_3, l_4) \]

\[ \langle \Theta(l_1)\Theta(l_2)\Theta(l_3)\Theta(l_4) \rangle_C = \frac{1}{2} \int \frac{d^2L}{(2\pi)^2} \delta(l_1 + l_2 + L) \delta(l_3 + l_4 - L) T^{(\ell_1\ell_2)}_{(\ell_3\ell_4)}(L) + \text{perms.} \]
Equilateral \[ k_1 + k_2 + k_3 = 0, |k_1| = |k_2| = |k_3| \]

\[ T(k_1) + T(k_2) + T(k_3) = \begin{cases} b > 0 & \text{for } k_1 \neq 0, k_2 \neq 0, k_3 = 0 \\ b < 0 & \text{for } k_1 = 0, k_2 = 0, k_3 \neq 0 \end{cases} \]
Millennium simulation
Near-equilateral to flattened:

\[ k_1 \quad k_2 \quad k_3 \]

\[ b > 0 \quad b < 0 \]
Squeezed bispectrum is a *correlation* of small-scale power with large-scale modes

For more pretty pictures and trispectrum see: The Real Shape of Non-Gaussianities, arXiv:1107.5431
Squeezed bispectrum

‘Linear-short leg’ approximation very accurate for large scales where cosmic variance is large

\[ l_1 \ll l_2 \leq l_3 \] with modulation field(s) \( X_i \)

\[
\langle \tilde{T}_{l_1m_1} \tilde{T}_{l_2m_2} \tilde{T}_{l_3m_3} \rangle \approx C_{l_1}^{TX_i} \left\langle \frac{\delta}{\delta X_{i,l_1m_1}} \left( \tilde{T}_{l_2m_2} \tilde{T}_{l_3m_3} \right) \right\rangle
\]

Correlation of the modulation with the large-scale field

Response of the small-scale power to changes in the modulation field (non perturbative)

Note: uses only the linear short leg approximation, otherwise non-perturbatively exact

Example: local primordial non-Gaussianity

Primordial curvature perturbation is modulated as

\[ \zeta = [1 + (3f_{NL}/5)\zeta_g]\zeta_g \]

\[ l_1 \ll 100: \text{modulation super-horizon and constant through last-scattering, } \zeta_g = \zeta_0^* \]

\[ b_{l_1l_2l_3} \approx \frac{6}{5} f_{NL} C_{l_1}^{T\zeta_0^*} (\tilde{C}_{l_2} + \tilde{C}_{l_3}) \]
Even with $f_{NL} = 0$, we observe CMB at last scattering modulated by other perturbations.
What is the modulating effect of large-scale super-horizon perturbations?

Single-field inflation: only one degree of freedom, e.g. everything determined by local temperature (density) on super-horizon scales

Cannot locally observe super-horizon perturbations (to \( O(\frac{k^2}{H^2}) \))

Observers in different places on LSS will see statistically exactly the same thing (at given fixed temperature/time from hot big bang)
- local physics is identical in Hubble patches that differ only by super-horizon modes
BUT: a distant observer will see modulations due to the large modes <~ horizon size today - can see and compare multiple different Hubble patches

- Super-horizon modes induce linear perturbations on all scales
  linear CMB anisotropies on large scales \((l < 100)\)

- Sub-horizon perturbations are observed in perturbed universe:
  - small-scale perturbations are modulated by the effect large-scale modes
  squeezed-shape non-Gaussianities
Linear CMB anisotropies

Linear perturbation theory with
\[ ds^2 = a(\eta)^2 [(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)dx^2] \]

Using the geodesic equation in the Conformal Newtonian Gauge:

\[
E(\eta_0) = a(\eta)E(\eta) \left[ 1 + \Psi(\eta) - \Psi_0 + \int_{\eta}^{\eta_0} d\eta (\Psi' + \Phi') \right]
\]

All photons redshift the same way, so \( kT \sim E \).

Recombination fairly sharp at background time \( \eta_*: \sim \text{constant temperature} \) surface. Also add Doppler effect:

\[
T(\hat{n}, \eta_0) = (a_* + \delta a)T_* \left[ 1 + \Psi(\eta_*) - \Psi_0 + \hat{n} \cdot (v_o - v) + \int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi') \right]
\]

\[
= T_0 \left[ 1 + \frac{\delta a}{a_*} + \Psi(\eta_*) - \Psi_0 + \hat{n} \cdot (v_o - v) + \int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi') \right]
\]
\[ \rho_{\gamma} \propto T^4 \propto a^4 \]

\[ \Rightarrow \quad \frac{\Delta T_0}{T}(\hat{n}) = \frac{\Delta \gamma(\eta_*)}{4} + \Psi(\eta_*) - \Psi_0 + \hat{n} \cdot (v_o - v) + \int_{\eta_*}^{\eta_0} d\eta (\Psi' + \Phi') \]

- Temperature perturbation at recombination
- Sachs-Wolfe
- Doppler
- ISW
Note: no scale on which Sachs-Wolfe $\Phi/3$ result is accurate. Doppler dominates at $l \sim 60$ because other terms cancel.
Alternative

\[ d\Phi = \frac{\partial \Phi}{\partial \eta} d\eta + \frac{\partial \Phi}{\partial \chi} d\chi \]

\[ a_A E_A = a(\eta) E(\eta) \left[ 1 + \Psi(\eta) - \Psi_A + \int_{\eta}^{\eta_A} d\eta \partial_\eta (\Psi + \Phi) \right] \]

\[ = a(\eta) E(\eta) \left[ 1 - \Phi(\eta) + \Phi_A + \int_{\eta}^{\eta_A} d\eta \partial_\chi (\Psi + \Phi) \right] \]

\[ \Delta T(\hat{n}) = \frac{\Delta \gamma}{4} - \Phi + \hat{n} \cdot (v_A - v) + \int_{\eta}^{\eta_A} d\eta \hat{n} \cdot \nabla (\Psi + \Phi) \]

\[ = \zeta_\gamma + \hat{n} \cdot (v_A - v) + \int_{\eta}^{\eta_A} d\eta \hat{n} \cdot \nabla (\Psi + \Phi) \]

Gauge-invariant 3-curvature on constant temperature hypersurfaces;
Redshifting from $\delta N$ expansion of the beam makes the $\delta N$ expansion from inflation observable
(but line of sight integral is larger on large-scales: overdensity looks colder)
Non-linear effect due to redshifting by large-scale modes?

Large-scale linear anisotropies are due to the linear anisotropic redshifting of the otherwise uniform (zero-order) temperature last scattering surface

\[ T \rightarrow (1 + \Delta T)T \]

Also non-linear effect due to the linear anisotropic redshifting of the linear last scattering surface

\[ \Delta T_{\text{small}} \rightarrow (1 + \Delta T_{\text{large}})\Delta T_{\text{small}} \]

Reduced bispectrum

\[ b_{l_1 l_2 l_3} \approx C_{l_1} \left( \tilde{C}_{l_2} + \tilde{C}_{l_3} \right) \]

Large-scale power spectrum

Small-scale (non-perturbative) power spectrum

(Actually very small, so not very important)
Linear effects of large-scale modes

- Redshifting as photons travel through perturbed universe

- Transverse directions also affected: perturbations at last scattering are distorted as well as anisotropically redshifted
Jabobi map relates observed angle to physical separation of pair of rays

\[ \xi_I(\lambda) = D_{IJ}(\lambda) \delta \theta_J \]

Physical separation vector orthogonal to ray

Jacobi map

Angular separation seen by observer at \( A \)

Evolution of Jacobi map:

\[ \frac{d^2 D_{IJ}}{d\lambda^2} = T_{IK} D_{KJ} \]

Optical tidal matrix depends on the Riemann tensor:

\[ T_{IJ} \equiv -E_i^b E_j^c k^a k^d R_{abcd} \]

\( (k^a \) is wave vector along ray, \( E_i^a \) projects into ray-orthogonal basis)
‘Riemann = Weyl + Ricci’

Non-local part (does not depend on local density):
- e.g. determined by Weyl (Newtonian) potential \( \frac{1}{2}(\Phi + \Psi) \)

differential deflection of light rays ⇒ convergence and shear of beam

- (Weyl) **lensing**

(can be modelled as transverse deflection angle)

Einstein equations relate Ricci to stress-energy tensor: depends on local density

⇒ ray area changes due to expansion of spacetime as the light propagates

- **Ricci focussing**

(can not be modelled by deflection angle)

FRW background universe has Weyl=0, Ricci gives standard angular diameter distance

At radial distance \( \chi_* \), trace of Jacobi map determines physical areas: \( \frac{D}{2} = \chi_* a_* \)
Beam propagation in a perturbed universe, e.g. Conformal Newtonian Gauge

\[ ds^2 = a^2(\eta)[(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j] \]

\[ \xi_I(\lambda) = \mathcal{D}_{I,J}(\lambda)\delta\theta_J \]

Trace-free part of Jacobi map depends on the shear:

\[ \gamma_{IJ} = \nabla_{(I} \nabla_{J)} \psi \]

\[ \psi \equiv -2 \int_0^{\chi_*} d\chi \frac{\chi_* - \chi}{\chi \chi_*} \Psi_W (\chi \hat{n}, \eta_A - \chi) \]

Area of beam determined by trace of Jacobi map:

\[ \mathcal{D}(\hat{n}, \eta)/2 = \chi(\hat{n}, \eta)a(\eta)[1 + \Phi_A - \Phi - \kappa + \hat{n} \cdot v_A] \]

CMB is constant temperature surface:

\[ \eta = \eta_* + \delta\eta \]

\[ \rho_\gamma \propto T^4 \propto a^4 \]

Radial displacement (small, \( \delta\chi \ll \chi_* \))

Ricci focussing

(Weyl) convergence

Local aberration
\[
\frac{D}{2} \approx \chi_* a_*(1 + \zeta_y - \kappa)
\]

Overdensity ($\zeta$ larger)

underdensity

Ricci focusing: beam contracts more leaving LSS ⇒ same physical size looks smaller

(Weyl lensing effect not shown and partly cancels area effect)
Gauge-invariant Ricci focussing $\zeta_\gamma \equiv \Delta_\gamma / 4 - \Phi$
Observable CMB bispectrum from single-field inflation

Linear-short leg approximation for nearly-squeezed shapes:

\[
\langle \tilde{T}_{l_1 m_1} \tilde{T}_{l_2 m_2} \tilde{T}_{l_3 m_3} \rangle \approx C_{l_1}^{TX_1} \left\langle \frac{\delta}{\delta X^*_{i,l_1 m_1}} (\tilde{T}_{l_2 m_2} \tilde{T}_{l_3 m_3}) \right\rangle
\]

Where \( X_i \) here is \( \delta T, \kappa \) and \( \zeta_\gamma \), with \( \frac{D}{2} \approx \chi_\kappa (1 + \zeta_\gamma - \kappa) \). For super-horizon adiabatic modes \( \zeta_\gamma = \zeta_0 \).

Weyl lensing bispectrum

\[
b_{l_1 l_2 l_3} = \frac{1}{2} [(l_1(l_1 + 1) + l_2(l_2 + 1) - l_3(l_3 + 1)] C_{l_1}^{T\psi} \tilde{C}_{l_2} + \text{perms}
\]

Squeezed limit (\( l_1 \ll l \))

\[
b_{l_1 l_2 l_3} \approx C_{l_1}^{T\kappa} \left[ \frac{1}{l^2} \frac{d(l^2 \tilde{C}_l)}{d \ln l} + \cos 2\phi_{l_1 l} \frac{d \tilde{C}_l}{d \ln l} \right]
\]

\( 1 \equiv (l_2 - l_3)/2 \)

Ricci focussing bispectrum

\[
b_{l_1 l_2 l_3} \approx C_{l_1}^{T\zeta_0} \frac{1}{2} \frac{d}{d \ln X^*} \left[ \tilde{C}_{l_2} + \tilde{C}_{l_3} \right]
\]

Squeezed limit (\( l_1 \ll l \))

\[
b_{l_1 l_2 l_3} \approx -C_{l_1}^{T\zeta_0} \frac{1}{l^2} \frac{d}{d \ln l} (l^2 \tilde{C}_l)
\]

+ anisotropi redshifting bispectrum (from before)

\[
b_{l_1 l_2 l_3} \approx C_{l_1} (\tilde{C}_{l_2} + \tilde{C}_{l_3})
\]
Weyl lensing bispectrum

\( C_l^{T\kappa} \): Correlation between lenses and CMB temperature?

\[
\Delta T_{\text{ISW}}(\hat{n}) = 2 \int_0^{\chi^*} d\chi \dot{\Psi}(\chi \hat{n}; \eta_0 - \chi)
\]

Overdensity: magnification correlated with positive Integrated Sachs-Wolfe (net blueshift)

Underdensity: demagnification correlated with negative Integrated Sachs-Wolfe (net redshift)
$C_l^{T\kappa}$: Correlation between lenses and CMB temperature?

- The late Integrated Sachs Wolfe effect (late ISW) at low redshift from decaying potentials
- Large-scale modes that span recombination and also act as lenses
- The early Integrated Sachs Wolfe effect (early ISW) due to the transition from radiation to matter domination, and decaying modes
- Lenses close to last-scattering being correlated to density perturbations that have infall giving a Doppler signal in the CMB
- Doppler signal from scattering at reionization
- Lenses at last-scattering that directly correlate perturbations to lensing at the recombination surface
- Non-linear Rees-Sciama signal at low redshift from non-linear gravitational clustering
- Non-linear SZ signal from scattering in clusters
- Correlations due to foreground contaminants

Linear effects, All included in self-consistent linear calculation with CAMB

Non-linear growth effect - estimate using e.g. Halofit

Potentially important, but frequency dependent - ‘foregrounds’
Contributions to the lensing-CMB cross-correlation, $C^T\psi_l$

(note Rees-Sciama contribution is small, numerical problem with much larger result of Verde et al, Mangilli et al.; see also Junk et al. 2012 who agree with me)
Weyl lensing total + Ricci focusing (+ estimates of sub-horizon dynamics)
Does this look like squeezed non-Gaussianity $f_{NL}$ from multi-field inflation (local modulation of small scale perturbation amplitudes in each Hubble patch)?

<table>
<thead>
<tr>
<th>Data used</th>
<th>$\sigma_{f_{NL}}$</th>
<th>Weyl</th>
<th>Ricci</th>
<th>Redshift</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>4.3</td>
<td>9.5</td>
<td>1.5</td>
<td>-0.22</td>
<td>10.7</td>
</tr>
<tr>
<td>Planck T</td>
<td>5.9</td>
<td>6.4</td>
<td>1.0</td>
<td>-0.22</td>
<td>7.1</td>
</tr>
<tr>
<td>T ($l_1 &lt; 60$)</td>
<td>4.6</td>
<td>10.6</td>
<td>1.7</td>
<td>-0.25</td>
<td>12.0</td>
</tr>
<tr>
<td>Planck T ($l_1 &lt; 60$)</td>
<td>6.2</td>
<td>7.0</td>
<td>1.1</td>
<td>-0.25</td>
<td>7.9</td>
</tr>
<tr>
<td>T+E</td>
<td>2.1</td>
<td>2.6</td>
<td>1.1</td>
<td>-0.05</td>
<td>3.7</td>
</tr>
<tr>
<td>Planck T+E</td>
<td>5.2</td>
<td>4.3</td>
<td>1.0</td>
<td>-0.15</td>
<td>5.2</td>
</tr>
</tbody>
</table>

Dominated by lensing $f_{NL} \sim 6 - 10$

Ricci is an $O(1)$ correction

Calculation reliable for $l_1 < 60$ where dynamical effects suppressed by small $k^2/H^2$: do not need fully non-linear dynamical calculation of bispectrum a la Pitrou et al to make reliable $f_{NL}$ constraint

*TABLE I: Individual and total biases on primordial local-model non-Gaussianity parameterized by $f_{NL}$ for CMB temperature and $E$-polarization data with Planck-like noise (assuming isotropic coverage over the full sky with sensitivity $\Delta T = \Delta Q/2 = \Delta U/2 = 50 \mu K\, \text{arcmin}$ [$N_l^T = N_l^E / 4 = 2 \times 10^{-4} \mu K^2$] and a beam FWHM of 7 arcmin) or cosmic-variance limited data with $l_{max} = 2000$. Results are assuming that non-$f_{NL}$ contributions are only significant at $l_1 \leq 300$ and negligible dynamical effects; the $l_1 < 60$ results are filtered to only use large scale modulations and are therefore immune to small-scale modulation effects. The bias is the systematic error on $f_{NL}$ if the given contribution is neglected, which can be compared to $\sigma_{f_{NL}}$ which is the Fisher error estimate (including lensing signal variance).*
Signal easily modelled
Squeezed shape but different phase, angle and scale dependence

Lensing

$f_{NL}$

Lewis, Challinor, Hanson 1101.2234
Note: ‘Maldacena’ bispectrum

\[
(\tilde{\zeta}(k_1)\tilde{\zeta}(k_2)\tilde{\zeta}(k_3)) \approx -\frac{1}{(2\pi)^{3/2}} \delta(k_1 + k_2 + k_3) P_\zeta(k_1) \frac{1}{2} \left[ \frac{1}{k_3^3} \frac{d}{d \ln k_3} (k_3^3 P_\zeta(k_3)) + \frac{1}{k_2^3} \frac{d}{d \ln k_2} (k_2^3 P_\zeta(k_2)) \right]
\]

Consistency relation: \( f_{NL} \sim O(n_s - 1) \)

is not an observable
- cannot measure comoving curvature perturbations on scales larger than the horizon directly
- \( d/d(\ln k) \) and CMB transfer functions do not commute: cannot get correct result from primordial \( f_{NL} \sim (n_s - 1) \)

Observable CMB analogue is Ricci focussing bispectrum

\[ b_{l_1 l_2 l_3} \approx -C_{l_1}^{T \zeta \zeta} \frac{1}{l^2} \frac{d}{d \ln l} (l^2 \tilde{C}_l) \]

- larger because of acoustic oscillations, non-zero for \( n_s = 1 \)
- different shape to \( f_{NL} \) in CMB, but projects as \( f_{NL} = O(1) \)

Question: primordial bispectrum calculations includes time shift \( \frac{d}{d \ln k} \) terms
- not correct to calculate effective \( f_{NL} \) at end of inflation, what to do? (e.g. features)
Lensing of primordial non-Gaussianity

General case at leading order: Hanson et al. arXiv:0905.4732

Bispectrum slices are smoothed by lensing, just like power spectrum

FIG. 5: The fractional change in the reduced bispectrum slice $b_{10,l,l+10}$ due to lensing. The blue line shows the non-perturbative approximation of this paper, the black line shows the leading-order perturbative result from Ref. [1]. The red lines show the result of 1000 Monte Carlo simulations of Ref. [1] smoothed over $\Delta l = 5$. The new approximation only needs to lens the isotropic component of the bispectrum, and then is both significantly more accurate on small scales and faster to compute.

BUT lensing preserves total power: expect $\sim 0$ bias on primordial $f_{NL}$ estimators
Squeezed trispectrum

• Lensing gives large trispectrum, this is what is used for lensing reconstruction

• Also want to look for primordial trispectrum

e.g. from primordial modulation \( \zeta(x) = \zeta_0(x)[1 + \phi(x)] \)

Squeezed shape, constant modulation \( \Rightarrow T(\hat{n}) \approx T_g(\hat{n})[1 + \phi(\hat{n}, r_*)] \).

Easy accurate estimator for \( \tau_{NL} \) is \( \tau_{NL}(L) \equiv \frac{C_L^\phi}{C_L^{\zeta_*}} \)

\[ \hat{\tau}_{NL} \approx L_{min}^2 \sum_{L=L_{min}}^{\infty} \frac{2L+1}{L^2(L+1)^2} \frac{\hat{C}_L^\phi}{C_L^{\zeta_*}} \] (optimal to percent level)

Lensing bias on \( \tau_{NL} \)

\[ \langle \hat{\tau}_{NL} \rangle \approx L_{min}^2 \sum_{L=L_{min}}^{\infty} \frac{2L+1}{L^2(L+1)^2} \frac{\alpha_L^2 C_L^{\kappa}}{C_L^{\zeta}} \]

All \( \tau_{NL} \) signal at low \( l < \sim 10 \): cut to avoid blue lensing signal at higher \( l \)

Then fairly small, 17 to 40 depending on data: small compared to \( \sigma_{\tau_{NL}} \geq 150 \)

Lensing not a problem for \( \tau_{NL} \) constraints (because they are so weak!)
Conclusions

- Single field inflation predicts significant non-Gaussianity in the observed CMB
  - mostly due to (Weyl) lensing
  - total projects onto $f_{NL} \sim 7$ for Planck temperature
  - Ricci focussing expansion of beam recovers the $\delta N$ from inflation,
    : gives equivalent of consistency relation, but larger
    : small and not quite observable, projects on to $f_{NL} \sim 1$
  - Squeezed calculation reliable at $l_1 < 60$
    : robust constraints on $f_{NL}$ without 2\textsuperscript{nd} order dynamics
  - effect on trispectrum is small

- Lensing bispectrum signal important but distinctive shape
  - dominated by late ISW correlation, but other term important (eg. early ISW)
  - predicted accurately by linear theory (Rees-Sciama is tiny)

- On smaller scales, and non-squeezed shapes, need full numerical calculation
  of non-linear dynamical effects in CMB

Question: for numerical calculation of squeezed non-linear effects, how to you handle/separate the large lensing signal? ($f_{NL} \sim 3$ sounds like mostly lensing to me)