Stats Examples: Answers 2

1. Let X be the number that are too long, then $X \sim B(10, 0.1)$. Hence

$$P(X < 2) = P(X = 0) + P(X = 1) = C_0^{10} 0.1^0 (1 - 0.1)^{10} + C_1^{10} 0.1^1 (1 - 0.1)^9 = 0.9^{10} + 10 \times 0.1 \times 0.9^9 \approx 0.736.$$

2. There are 800 components, with 0.001 probability of failure each, so the expected number of failures is $\lambda = 800 \times 0.001 = 0.8$. Let X = number of components that fail.

i $P(X = 0) = e^{-\lambda} = 0.449$, $P(X = 1) = e^{-\lambda}\lambda = 0.359$, $P(X = 2) = e^{-\lambda}\lambda^2/2 = 0.144$. $P(X > 2) = 1 - P(X \leq 2) = 0.0474$.

ii $P(\mathrm{OK}) = \mathrm{P(X = 0) + 0.5P(X = 1) + 0.2P(X = 2)} = 0.6573$.

3. Let X = number of damaged chocolates in a box.

i $X \sim B(30, 0.05)$.

$$\begin{aligned} P(X > 2) = 1 - P(X \leq 2) &= 1 - P(2) - P(1) - P(0) \\ &= 1 - C_2^{30} p^2 (1-p)^{28} - C_1^{30} p (1-p)^{29} - (1-p)^{30} \\ &= 1 - \frac{30 \times 29}{2} 0.05^2 \times 0.95^{28} - 30 \times 0.05 \times 0.95^{29} - 0.95^{30} = 0.188. \end{aligned}$$

ii Approximate with Poisson, $\lambda = 0.05 \times 30 = 3/2$.
Now $P(X > 2) = 1 - P(2) - P(1) - P(0) = 1 - \left(\lambda^2/2 + \lambda + 1\right) e^{-\lambda} = 0.191$.

4. Model the arrival of calls as a Poisson process with a rate $\nu = 20/\mathrm{hr}$. In 2 minutes the expected number is $\lambda = (2/60) \times 20 = 2/3$. Let X = number of calls, then $P(X = 0) = e^{-\lambda} = 0.513$.
If the downtime is t minutes then $\lambda = 20t/60 = t/3$ and we want the probability to exceed 0.95, so

$$e^{-\lambda} > 0.95 \implies -\lambda > \ln(0.95) \implies t/3 < -\ln(0.95) \implies t < 0.154\mathrm{min}.$$

i.e. about 9 seconds.

5.

$$\int x^3 dx = \frac{x^4}{4} + \mathrm{const}$$

$$\int_1^3 \frac{dx}{x^2} = \left[\frac{-1}{x}\right]_1^3 = \frac{-1}{3} - (-1) = \frac{2}{3}$$

$$\int_0^\infty e^{-2x} dx = \left[-\frac{e^{-2x}}{2}\right]_0^\infty = 0 + \frac{e^0}{2} = \frac{1}{2}$$

Finally integrating by parts

$$\int_0^1 x e^{x/2} = \left[x 2 e^{x/2}\right]_0^1 - \int_0^1 2 e^{x/2} dx = 2e^{1/2} - \left[4 e^{x/2}\right]_0^1 = 2e^{1/2} - 4e^{1/2} + 4 = 4 - 2\sqrt{e} \approx 0.7026.$$

6.

i $\int_{-\infty}^{\infty} f(x)dx = \int_0^{0.1} k(0.1 - x)dx = k\left[0.1x - x^2/2\right]_0^{0.1} = k(0.1^2 - 0.1^2/2) = 0.005k \implies k = 200$.

ii $P(0.02 < X < 0.06) = \int_{0.02}^{0.06} (20 - 200x)dx = \left[20x - 100x^2\right]_{0.02}^{0.06} = 0.84 - 0.36 = 0.48$.

iii $\langle X \rangle = \int x f(x) dx = \int_0^{0.1} (20x - 200x^2)dx = \left[10x^2 - 200x^3/3\right]_0^{0.1} = 0.0333$.
$\langle X^2 \rangle = \int x^2 f(x) dx = \int_0^{0.1} (20x^2 - 200x^3)dx = \left[20x^3/3 - 50x^4\right]_0^{0.1} = 0.00167$
$\implies \mathrm{var(X)} = \langle \mathrm{X}^2 \rangle - \langle \mathrm{X} \rangle^2 \approx 5.6 \times 10^{-4}$.

7. There are two possibilities A =action, and A^c =no action. The expected loss if action is taken is 2% GBP, whether the skeptic is correct or not. If action is not taken the expected loss is (from the rule for the expected value of a function of a random variable)

$$\text{loss if correct} \times P(\text{correct}) + \text{loss if incorrect} \times P(\text{incorrect}) = 0\% \times 0.7 + 20\% \times (1 - 0.7) = 20\% \times 0.3 = 6\%,$$

hence an expected loss of 6% if no action is taken. Since $6\% > 2\%$ the skeptic can help to minimize expected loss by supporting action.

8. This is an exponential distribution, so the answers are standard. Explicitly for $\nu = 5$ the mean is

$$\langle T \rangle = \int_0^\infty 5te^{-5t}dt = \left[-te^{-5t}\right]_0^\infty + \int_0^\infty e^{-5t}dt = \left[-e^{-5t}/5\right]_0^\infty = 1/5.$$

The median x has

$$\int_0^x 5e^{-5t}dt = 1/2 \implies \left[-e^{-5t}\right]_0^x = 1 - e^{-5x} = 1/2 \implies e^{-5x} = 1/2 \implies x = -\log(1/2)/5 \approx 0.139.$$

To find the variance we get

$$\langle T^2 \rangle = \int_0^\infty 5t^2e^{-5t}dt = [0] + \int_0^\infty 2te^{-5t}dt = [0] + \int_0^\infty \frac{2}{5}e^{-5t}dt = \left[-e^{-5t}2/5^2\right]_0^\infty = 2/25$$

and hence

$$\implies \sigma_T = \sqrt{\langle T^2 \rangle - \langle T \rangle^2} = 1/5.$$

$$P(T > 1s) = \int_1^\infty 5e^{-5t} = \left[-e^{-5t}\right]_1^\infty = e^{-5} \approx 0.00674$$