Stats Examples: Answers 1

1. P(Exactly 10) is probability of getting 9 OK and then 10th bad. Since each is independent this is

$$P(\text{9OK})\text{P(1bad)} = (1-0.15)^9 \times 0.15 = 0.0347.$$

P(10 or more) is $(1-0.15)^9 = 0.232$.

2. Let T = positive test result, C = has cancer. Use Bayes theorem to write what we want as

$$P(C|T) = \frac{P(T|C)P(C)}{P(T)}$$

We are told that $P(T|C) = 0.95$, $P(T|\text{not C}) = 0.1$ and $P(C) = 0.01$. We get $P(T)$ using

$$P(T) = P(T|C)P(C) + P(T|\text{notC})\text{P(not C)} = 0.95 \times 0.01 + 0.1 \times 0.99 = 0.1085$$

hence

$$P(C|T) = \frac{0.95 \times 0.01}{0.1085} = 0.0876$$

(3 significant figures).

Comment: This shows that if you are going to do random testing for rare events, the rate of false positives has to be very low for a positive test result not to be most likely a false positive. Such tests are only likely to be useful with a non-random sample (e.g. people who already report a symptom), or if there are follow-up tests that can be done.

3. Let $1S$ = 1 sent, $0S$ = 0 sent. We are told $P(1|0S) = 0.1$ and $P(0|1S) = 0.05$, $P(1S) = 0.6$.
i). $P(\text{accurate}) = \text{P(1|1S)P(1S)} + \text{P(0|0S)P(0S)} = 0.95 \times 0.6 + 0.9 \times 0.4 = 0.930$
ii). $P(1S|1) = P(1|1S)P(1S)/P(1) = 0.95 \times 0.6/(0.95 \times 0.6 + 0.1 \times 0.4) = 0.934$
iii). $P(0S|0) = P(0|0S)P(0S)/P(0) = 0.9 \times 0.4/(0.9 \times 0.4 + 0.05 \times 0.6) = 0.923$

Using triples $P(0|000)$ is probability of getting 2 or 3 zeros starting with three zeros, so

$$P(0|000) = 0.9^3 + 0.9^2 \times 0.1 \times C^3_2 = 0.972.$$

Similarly

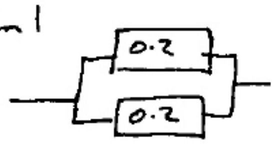
$$P(1|111) = 0.95^3 + 0.95^2 \times 0.05 \times C^3_2 = 0.99275.$$

Hence

$$P(\text{accurate}) = \text{P(0|000)P(000)} + \text{P(1|111)P(111)} = 0.972 \times 0.4 + 0.99275 \times 0.6 \approx 0.984.$$

4.

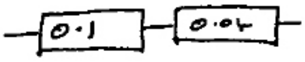
(i) Subsystem 1 (SS1)



$P(\text{SS1 fails}) = 0.2 \times 0.2 = 0.04$

$\therefore P(\text{SS1 does not fail}) = 0.96$

Subsystem 2 (SS2)

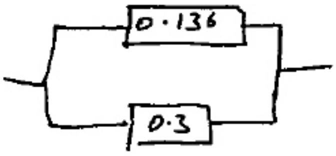


$P(\text{SS2 does not fail}) = (1-0.1)(1-0.04)$

$= 0.9 \times 0.96 = 0.864$

$P(\text{SS2 fails}) = 0.136$

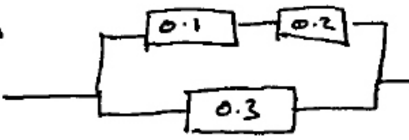
System 1



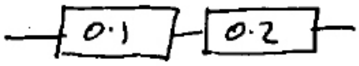
$P(\text{System fails}) = 0.136 \times 0.3 = 0.0408$

$P(\text{System doesn't fail}) = \underline{\underline{0.9592}}$

(ii) Given component * fails, the system becomes



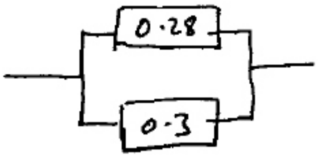
Subsystem A (SSA)



$P(\text{SSA does not fail}) = (1-0.1)(1-0.2) = 0.7$

$P(\text{SSA fails}) = 0.28$

System 2



$P(\text{System fails} \mid \text{Component * fails}) = 0.28 \times 0.3$

$= 0.084$

$\therefore P(\text{System does not fail} \mid \text{Component * fails}) = \underline{\underline{0.916}}$

iii $P(*\text{failed}|\text{OK}) = \text{P}(\text{OK}| * \text{failed})\text{P}(*\text{failed})/\text{P}(\text{OK}) = 0.916 \times 0.2/0.9592 \approx 0.191$.

5. The first sentence in the question is irrelevant. Note that $P(A \text{ and B}) \leq \text{P(A)}$ for any A and B (so in this case $P(a) \leq P(d)$). Also $P(A \text{ or B}) \geq \text{P(A)}$, so $P(b) > P(d)$.

Let G = John likes playing games, and S = John speaks Spanish. Now "not M" can be written $M^c = G \cap S$, so $P(c) = P(M^c \cup L) = P((G \cap S) \cup L)$, where L is winning the lottery (which is very low but non-zero probability). Hence $P(c) = P((G \cap S) \cup L) > P(G \cap S) = P(a)$.

So the ordering is $P(b) > P(d) > P(c) > P(a)$.

6.

a. Number of ways of picking 5 cards from 52 is $C_5^{52} = 52!/(47!5!) = 2598960$. So the probability of getting exactly this selection is $1/2598960 \approx 3.8 \times 10^{-7}$.

b. $39/52 \times 38/51 \times 37/50 \times 36/49 \times 35/48 \approx 0.221$.

c. P(1 or more sevens) = 1-P(no sevens) $= 1 - 48/52 \times 47/51 \times 46/50 \times 45/49 \times 44/48 \approx 0.341$.

d. Let S = spade, S^c= not spade, and we want exactly 2 spades out of five random cards. Probability of getting ordered sequence $S - S - S^c - S^c - S^c$ is $13/52 \times 12/51 \times 39/50 \times 38/49 \times 37/48 = 0.0274$ and there are C_2^5

ways to arrange two cards from five to be the spades, so the probability is $0.0274 \times 5!/(3!2!) \approx 0.274$.
Note that this cannot be obtained from the Binomial distribution because cards drawn without replacement are not independent (e.g. getting one spade reduces the probability that the next card is also a spade). The Binomial result is however approximately correct since 52 is quite a lot of cards, so the result with replacement is not that different $C^5_2 \times (13/52)^2 \times (1 - 13/52)^3 \approx 0.26$.

7. Let X = my choice is correct, with $P(X) = 1/3$. Let B = the information that one of the other pieces has been removed (which, by the constraints on how the paper is removed, will have nothing behind it, if X is true or not). Now Bayes' theorem gives

$$P(X|B) = \frac{P(B|X)P(X)}{P(B)}$$

with $P(X) = 1/3$, $P(B) = 1$ (since the rules are that B always happens), and $P(B|X) = 1$ as well, since the rules hold regardless of whether I am correct or not. Hence $P(X|B) = P(X) = 1/3$. So probability that I was correct is unchanged, so the note is under the other piece at the end with probability 2/3.
It may be helpful to enumerate possibilities
1. with probability 1/3, chose correct paper. At the end the other piece is incorrect, whichever is removed.
2. with probability 2/3, chose incorrect paper. The other piece of paper without the note is removed. At the end other piece is correct.
So probability the note is under the other piece at the end is 2/3.
This is commonly known as the 'Monty Hall problem', and is often a source of a lot of confusion! See `http://en.wikipedia.org/wiki/Monty_Hall_problem` if you doubt this answer.

In the case where a gust of wind happens to blow away a piece of paper which does not affect the game, things are different: the gust was more likely to have destroyed the game if your pick and the note were different pieces of paper, since in this case there was only a 1/3 chance that the gust removed the third piece. Explicitly let G = gust blows away paper with nothing behind it and the one you didn't pick, then using Bayes theorem with $P(G) = P(G|X)P(X) + P(G|X^c)P(X^c)$ gives

$$P(X|G) = \frac{P(G|X)P(X)}{P(G)} = \frac{2/3 \times 1/3}{2/3 \times 1/3 + 1/3 \times 2/3} = 1/2.$$

So the answer in this case is 1/2.