

Advanced Cosmology : Problem Sheet 3

Deadline: Week 11, Thursday 30th April at 12:00. Approximate marks in [] with total marked out of 82.

Hand in at the MPS School Office. Solutions submitted up 24 hours late will be attract a penalty of 5%; no solutions will be accepted more than 24 hours late.

1 Sound speed [15]

The sound speed can be taken to be $c_s^2 = dP/d\rho$.

- (i) Demonstrate, using their redshift evolution, that if one has a fluid of photons (γ) and baryons (b , approximated as being pressureless), then

$$c_s^2 = \frac{1}{3} \left(\frac{3}{4} \frac{\rho_b}{\rho_\gamma} + 1 \right)^{-1}.$$

- (ii) What is the sound speed at recombination (at redshift z_*)? [assume a flat universe with $\Omega_{b,0}h^2 = 0.02$, $z_* = 1000$, $T_0 = 2.7K$]
- (iii) The amplitude of the sub-horizon radiation perturbations at recombination ($\eta = \eta_*$) have an approximately sinusoidal dependence in wavenumber k . If the baryon density is increased (keeping η_* and the background evolution fixed), would you expect the k -spacing of the oscillations to increase, stay the same, or decrease, and why?

2 Massive neutrinos [15]

Consider a flat universe ($\Omega_{tot} = 1$) filled with cold, pressureless matter and massive neutrinos (which make up a fraction $\Omega_\nu \ll 1$ of the critical density). The background dynamics of the universe in this situation can be approximated as matter dominated.

- (i) Write down the coupled system of equations for the cold matter and neutrino density perturbations on sub-horizon scales
- (ii) Show that, on scales smaller than the neutrino Jeans length (so that neutrino perturbations can be neglected compared to dark matter), perturbations in the remaining cold component grow as $\Delta_c \propto \eta^\alpha$, where $\alpha = (\sqrt{25 - 24\Omega_\nu} - 1)/2$. Hence find how Δ_c grows with time t .

3 Matter power spectrum [17]

1. Statistical homogeneity implies that the real-space correlation between a density at two points (at fixed time) is the same under a spatial translation, i.e. $\langle \Delta(\mathbf{x})\Delta(\mathbf{x}') \rangle = \langle \Delta(\mathbf{x} + \mathbf{c})\Delta(\mathbf{x}' + \mathbf{c}) \rangle$ where \mathbf{c} is any constant vector.
 - (a) Show that statistical homogeneity implies that $\langle \Delta(\mathbf{k})\Delta(\mathbf{k}') \rangle \propto \delta(\mathbf{k} + \mathbf{k}')$.
 - (b) What additional constraint does statistical isotropy impose?
2.
 - (a) Sketch the expected shape of the linear-theory matter power spectrum as a function of $\log(k)$
 - (b) Explain in your own words why it has the rough shape that it does.
 - (c) Indicate how the spectrum would change if n_s , the spectral index of the primordial curvature power spectrum is increased.
3. It is often useful to consider the density perturbation smoothed over some scale, for example a Gaussian smoothing with some scale σ :

$$\Delta_\sigma(\mathbf{x}') \equiv \int d^3\mathbf{x} \frac{e^{-(\mathbf{x}-\mathbf{x}')^2/2\sigma^2}}{(2\pi\sigma)^{3/2}} \Delta(\mathbf{x}). \quad (1)$$

- (a) Calculate the power spectrum P_{Δ_σ} of Δ_σ in terms of P_Δ [*hint*: convolution theorem].
- (b) Sketch roughly the expected form of P_{Δ_σ} compared to P_Δ .

4 Growth factor and velocity power spectrum [12]

On scales well within the horizon but larger than the Jeans length, at a given time we can define a (k -independent) growth factor g where

$$g \equiv \frac{d(\ln \Delta_m)}{d(\ln a)}. \quad (2)$$

- (i)
 - (a) What is g during matter domination?
 - (b) What is g in the far future if Λ (cosmological constant) then dominates?
- (ii) Calculate the power spectrum P_θ of $\theta \equiv \nabla \cdot \mathbf{v}_m / \mathcal{H}$ in terms of g and P_{Δ_m} .

5 Photon redshift in a perturbed universe [23]

Consider the propagation of photons (null geodesics) in a perturbed flat universe in the conformal Newtonian gauge (CNG) with metric

$$ds^2 = a(\eta)^2 [(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)d\mathbf{x}^2].$$

Normalize the affine parameter λ so that a photon's 4-momentum is given by $p^\mu = dx^\mu/d\lambda$. Working to linear order in the perturbations Ψ and Φ :

1. If an observer at rest in the CNG has 4-velocity $u^\nu = (1 - \Psi)\delta_{0\nu}/a$ show that $u_\nu = a(1 + \Psi)\delta_{0\nu}$ and hence that $u_\nu u^\nu = 1$. (remember we are only working to first-order in Ψ)
2. Show that if a photon with 4-momentum p^μ has an energy $E = p^\nu u_\nu$ for an observer with 4-velocity u^ν then

$$p^0 = \frac{d\eta}{d\lambda} = E(1 - \Psi)/a.$$

3. Calculate the $\Gamma_{\mu\nu}^0$ Christoffel symbols, and show hence that the 0 component of the geodesic equation can be written

$$\frac{dp^0}{d\lambda} + \left(\frac{a'}{a} + \Psi'\right)p^0 p^0 + 2p^0 p^i \frac{\partial\Psi}{\partial x^i} - \left(\frac{a'}{a}(1 - 2\Psi) - \Phi'\right)\frac{g_{ij}}{a^2}p^i p^j = 0$$

where i, j run over the spatial indices and primes denote conformal time derivatives. Using $p^i \partial/\partial x^i = d/d\lambda - p^0 \partial/\partial \eta$ then show this can be written as

$$\frac{d[aE(1 + \Psi)]}{d\lambda} = E^2(\Psi' + \Phi')$$

(*hint*: use the fact that $\epsilon \equiv aE$ is constant to zeroth order, writing p^0 in terms of ϵ).

4. Hence if the energy observed by a observer (at rest in the CNG) today is E_0 at conformal time η_0 , show that

$$E(\eta) = \frac{E_0}{a(\eta)} \left[1 + \Psi_0 - \Psi(\eta) - \int_{\eta}^{\eta_0} d\eta (\Psi' + \Phi') \right]$$

where the integral is along the photon path. Explain the relevance of this result for the observed CMB anisotropies.