

Early Universe : Problem Sheet 2

Deadline: Week 9, Monday 27th March, at 12:00. Approximate marks in [].

Hand in at the MPS School Office. Solutions submitted up 24 hours late will be attract a penalty of 5%; no solutions will be accepted more than 24 hours late.

1 Scalar field Lagrangian [12]

The Lagrangian for a homogeneous scalar field is

$$\mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

Derive the field equation using the Euler-Lagrange equations with the action $\int d^4x \sqrt{-g} \mathcal{L}_\phi$ in a flat FRW universe, and show that it is consistent with the density evolution equation. Change variables to conformal time to show that

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_{,\phi} = 0,$$

where a prime denotes $d/d\eta$, η is the conformal time and $\mathcal{H} = aH$.

2 Power law inflation [20]

An inflation model *power law inflation* has a potential given by

$$V(\phi) = V_0 \exp \left\{ -\sqrt{\frac{2}{\lambda}} \frac{\phi}{M_P} \right\},$$

where V_0 and λ are a priori free parameters of the model. Assume that the universe is flat and that the energy density is dominated by the inflaton.

- (i) Write down the relations for the density and pressure (ρ and P), the equation of motion and the slow-roll equations.
- (ii) Using the slow-roll approximation compute solutions for $\phi(t)$, $w(t)$, $a(t)$ and the slow-roll parameters ϵ_V and η_V . When does inflation end?
- (iii) What conditions have to be satisfied for “successful” inflation in this model?

3 Geodesics and the FRW solution [23]

This question is about geodesics, and then using a trick to calculate the Christoffel symbols and hence the Friedmann equations. Geodesics are paths through spacetime with extremal ‘length’, i.e. they extremize

$$S = \int ds = \int d\lambda \frac{ds}{d\lambda} = \int d\lambda L$$

where from the metric

$$L \equiv \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$

1. For a timelike geodesic with $\lambda = \tau$ the proper time, show that $x^\alpha(\tau)$ that extremizes S also extremizes

$$S_2 = \int d\tau L_2$$

with $L_2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$. Do we expect this to be a minimum or maximum?

2. Apply the Euler-Lagrange equations

$$\frac{d}{d\tau} \left(\frac{\partial L_2}{\partial(dx^\alpha/d\tau)} \right) = \frac{\partial L_2}{\partial x^\alpha}$$

to show that the general solution for the the path $x^\alpha(\tau)$ along the geodesic is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0.$$

3. Write down L_2 in the flat FRW metric

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j$$

and then apply the Euler-Lagrange equations with $x^\alpha = t$ find an equation for $d^2 t/d\tau^2$. By comparing this with Eq. 2 read off all the components α, β of $\Gamma_{\alpha\beta}^t$. (this is a handy trick for quickly calculating several Christoffel symbols)

4. Similarly calculate all the other non-zero Christoffel symbols. Explain why the Ricci tensor is expected to be diagonal with $R_{ij} \propto \delta_{ij}$, and calculate its non-zero components.
5. Calculate the Ricci scalar, and hence the Einstein equations for the FRW metric with the energy-momentum tensor

$$T_\mu^\nu = \text{diag} (\rho, -P, -P, -P).$$

4 Power spectrum from quantum fluctuations [20]

A scalar field perturbation operator in harmonic space $\hat{\delta\phi}$ is expanded as

$$\hat{\delta\phi}(\mathbf{k}, \eta) = w(k, \eta)\hat{a}(\mathbf{k}) + w^*(k, \eta)\hat{a}^\dagger(-\mathbf{k}),$$

where $w(k, \eta)$ is a solution of the field equation

$$w'' + 2\mathcal{H}w' + k^2w = 0.$$

- (i) By expanding $\hat{\delta\phi}$ and $\hat{\pi}_{\delta\phi}$ into Fourier modes show that

$$[\hat{\delta\phi}(\mathbf{x}, \eta), \hat{\pi}_{\delta\phi}(\mathbf{x}', \eta)] = i\delta(\mathbf{x} - \mathbf{x}')$$

if we require that

$$a^2(ww^{*'} - w'w^*) = i$$

and

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'), \quad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = 0 \quad [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0.$$

- (ii) If the Hubble rate H is constant solve for the comoving Hubble rate \mathcal{H} as a function of conformal time (with $-\infty < \eta < 0$).
- (iii) If H is constant show that

$$w(k, \eta) = \frac{H(k\eta - i)}{\sqrt{2k^3}} e^{-ik\eta}$$

solves the field equation and the normalization constraint. Sketch $|w|$ as a function of conformal time around the time a given k leaves the horizon.

5 Scales and times [13]

1. Assume naively that inflation happened with $H \sim H_i \sim \text{constant}$ and ended suddenly giving a hot big bang (radiation dominated universe) that started with $H = H_i$ when $a \sim 10^{-15}$. Does inflation increase or decrease the age of the universe between scale factor $a \sim 10^{-40}$ and $a \sim 10^{-15}$, and by roughly how much? (compared to a pure hot big bang model starting at $a = 0$)
2. Perturbations undergo different evolution depending on whether their (comoving) wavenumber k is smaller or larger than the (comoving) Hubble scale $\mathcal{H} = aH$.
 - (i) Compute the evolution of the comoving Hubble length with time during inflation ($w \approx -1$), radiation domination ($w = 1/3$) and matter domination ($w = 0$).
 - (ii) Sketch the behavior of the comoving Hubble length with time (perhaps using logarithmic scales). What does this tell us about the evolution of a perturbation with fixed comoving wavenumber, k ?

6 ϕ^4 inflation [12]

Consider the Inflation potential $V = \lambda\phi^4$, where λ is the self coupling. Assume that the field rolls towards $\phi = 0$ from the positive side.

- (i) Calculate the value of ϕ where each of the slow-roll conditions break down.
- (ii) Assuming that inflation ends when $\epsilon = 1$, calculate the number of e-folds of inflation that occur for an initial value of ϕ_i .
- (iii) Assuming that the scales of interest left the horizon 55 e-foldings before the end of inflation, calculate the scalar amplitude A_s and spectral index. Use the Planck result for A_s to put an order of magnitude value on λ .