# Early Universe : Problem Sheet 1

Deadline: Week 5, Monday 27th Feb, at 12:00. Approximate marks in []. Hand in at the MPS School Office. Solutions submitted up 24 hours late will be attract a penalty of 5%; no solutions will be accepted more than 24 hours late.

Qs 1 is relativity revision (related to cosmology), Q6 is cosmology revision; you could start work on these immediately.

## 1 The spatial curvature [17]

Calculate the 3-curvature, the Ricci scalar of the spatial part of the metric at fixed time:

$$dl^2 = a^2 \left[ \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right].$$

(first calculate the Christoffel symbols, then the curvature tensors). Note that at fixed time, there are no time derivatives.

## 2 Extra relativistic species [20]

According to the standard assumptions, there are three species of (massless) neutrinos. In the temperature range of 1MeV $< T <$ 100MeV, the density of the universe is believed to have been dominated by the black-body radiation of photons, electron-positron pairs, and three neutrinos all of which were in thermal equilibrium.

1. Neglecting any change in the degrees of freedom at $T >$ 100MeV, show using the Friedmann equation for a flat radiation-dominated universe $H^2 = 8\pi G\rho_R/3$ that for temperatures $T > 1MeV$ the time since the start of the hot big bang is given by

   $$t(T) = \left( \frac{A}{g_*} \right)^{1/2} \frac{M_P}{T^2}$$

   where $M_P \equiv \sqrt{\hbar c/(8\pi G)}$ is the reduced Planck mass, and $A$ is a constant that you should give explicitly. What is $g_*$? Put in $\hbar$, $c$ and $k_B$ factors give a result in standard rather than natural units. How long did it take from the big bang for the temperature to fall to $T =$ 1MeV? [Give the result in seconds]. [7,2]

2. How much time would it have taken if there were one other species of massless neutrino, in addition to the three which we are currently assuming? [3]

3. What would be the density of the universe (in $kg/m^3$ units) when $T =$ 1MeV under the standard assumptions, and what would it be if there were one other species of massless neutrino? What is the temperature in Kelvin at $T =$ 1MeV, and what is the redshift? [4,3]

4. What approximation have you made about the electrons and positron velocities, and is it reasonable? [1]

# 3 Freeze out of muons [16]

Muons $\mu^-$ are essentially identical to electrons, except that they are heavier ($m_\mu = 106MeV$); other than that, they also have the same charge and spin as the electron, and there is an antimuon $\mu^+$ analogous to the positron.

1. What is the value of the effective $g_*$ for muons when they are relativistic? [2]
2. When the temperature $T$ is a little above 106MeV, what particles besides the muons are contained in the thermal radiation that fills the universe? What is the total effective $g_*$? [5]
3. As $T$ falls below 106MeV, the muons disappear from the thermal equilibrium radiation. At these temperatures all of the other particles in the black-body radiation are interacting fast enough to maintain equilibrium, so the heat given off from the muons is shared among all the other particles. Letting $a$ denote the FRW scale factor, by what factor does the quantity $aT$ increase when the muons disappear? [In case you worry about it, ignore pions and QCD] [9]

# 4 CMB blackbody and $\mu$-distortions [16]

The distribution function of photons in a homogeneous and isotropic photon gas in kinetic equilibrium is

$$f_\gamma(E,T) = \frac{2}{(2\pi)^3}\frac{1}{e^{(E-\mu_\gamma)/T}-1}.$$

Consider the homogenous universe well after electron-positron annihilation is complete and ignore the very small effect of baryons.

1. At high temperatures $T \gg T_c \sim 0.5$keV double Compton scattering ($e^- + \gamma \leftrightarrow e^- + \gamma + \gamma$) happens frequently in equilibrium. In this case explain why $\mu_\gamma = 0$. [2]
2. At lower temperatures $T \ll T_c$ double Compton scattering no longer happens, and in general $\mu_\gamma$ can be non-zero. As the gas cools below $T_c$ it initially maintains its thermal distribution with $\mu_\gamma = 0$. If a small amount of energy is then injected into the photon gas to give an increase in the energy density by $\epsilon$, show by doing a first order series expansion in $\delta T$ and $\mu_\gamma$ that after (rapid) kinetic thermalization

$$\epsilon \approx \frac{T^4}{\pi^2}\int_0^\infty \frac{e^x x^3 dx}{(e^x-1)^2}\left[\frac{\mu_\gamma}{T} + x\frac{\delta T}{T}\right],$$

   where the temperature is changed by $\delta T$ and the chemical potential is changed by $\mu_\gamma$ (from zero). You can assume that $|\mu_\gamma/T| \ll 1$, $|\delta T/T| \ll 1$. [7]
3. If the energy injection increases the energy density *without* changing the number density of photons $n_\gamma$, show that after kinetic thermalization

$$\frac{\mu_\gamma}{T} \approx \frac{\epsilon}{\rho_\gamma}\frac{CX_3}{X_3^2 - X_2X_4}$$

where $X_k$ and $C$ are defined to be the values of the integrals

$$X_k \equiv \int_0^\infty \frac{x^k e^x dx}{(e^x - 1)^2} \qquad C \equiv \int_0^\infty \frac{x^3 dx}{e^x - 1}.$$

[This shows that processes depositing energy at $T < 0.5\text{keV}$ can give rise to a "$\mu$-distortion" in the CMB, i.e. a not-exactly blackbody spectrum.] [7]

# 5 Neutrino mass [16]

At least two neutrinos are thought to have a small mass, but small enough that in the early universe the neutrinos are still very relativistic. Assuming zero neutrino chemical potential:

1. The equilibrium distribution function at temperature $T$ for a single neutrino species in the limit in which the mass can be neglected, using natural units where $k_B = c = \hbar = 1$, is

$$f_\nu(p, T) = \frac{g_\nu}{(2\pi)^3} \frac{1}{e^{p/T} + 1}$$

   What is the meaning of $f_\nu$ and $p$ here, and what is the value of $g_\nu$? [2]

2. After a massive neutrino has completely decoupled at temperature $T_D$ and scale factor $a_D$, show that the energy density in these neutrinos is given by

$$\rho_\nu = \frac{T_\nu^4}{\pi^2} \int_0^\infty \frac{x^2 dx \sqrt{m_\nu^2/T_\nu^2 + x^2}}{e^x + 1}$$

   where $T_\nu \equiv T_D a_D / a$ [assume the neutrino was highly relativistic when it decoupled, so $m_\nu/T_D \ll 1$ is negligible]. [5]

3. By considering a series expansion for small $m_\nu/T_\nu$ show that if there is a massless neutrino with energy density $\rho_{\nu 0}$, for a nearly-relativistic massive neutrino with mass $m_\nu$

$$\rho_\nu \approx \rho_{\nu 0} \left(1 + \frac{5}{7\pi^2} \frac{m_\nu^2}{T_\nu^2}\right)$$

   to leading order in $m_\nu/T_\nu$. You can assume all the neutrinos were in thermal equilibrium before they decoupled. [6]

4. The effect of massive neutrinos can be seen in the linear CMB anisotropies if they are massive enough to affect the background evolution before recombination, e.g. when $\rho_\nu$ is significantly larger than $\rho_{\nu 0}$. Approximately what is the lightest neutrino (in electron volts) that has an observable effect on the linear CMB anisotropies? [3]

# 6 Age of the universe and time of recombination [15]

Consider a flat FRW model containing pressureless matter and a positive cosmological constant, with $\Omega_m + \Omega_\Lambda = 1$, $\Omega_\Lambda > 0$. Neglect any effects from the radiation energy density and pressure until part 5. Recall that if $a = 1$ today a solution for the scale factor is given by

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \left(\sinh\left[\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right]\right)^{2/3} .$$

1. Give a general result for the age of the universe today. What is the age of the universe (in billions of years) for $\Omega_\Lambda = 0.7$, $\Omega_m = 0.3$, $H_0 = 72\mathrm{km\,s^{-1}\,Mpc^{-1}}$? [2]

2. Given that the temperature of the CMB today is 2.725K, what was the scale factor when the photon temperature was 3000K at recombination? How old was the universe at recombination using the same parameters as Part 1? Given that we have neglected radiation, should this be an overestimate or underestimate? [3]

3. Show that $a(t) \approx Ct^{2/3}$ for $t \ll (\sqrt{\Omega_\Lambda} H_0)^{-1}$ and find the constant $C$. [2]

4. Using this approximation, what is the maximum comoving distance that radiation could have travelled from the big bang till recombination? Use the same parameters as Part 1 and give the result in Megaparsecs. [3]

5. At high redshifts ($z \gg 1$) the cosmological constant density should be negligible. Now accounting for radiation (but neglecting changes in $g_*$ for $T > 1\mathrm{MeV}$) show that for $z \gg 1$

$$H_0 t = \frac{2}{3\Omega_m^2}\left(\sqrt{\Omega_m a + \Omega_r}(\Omega_m a - 2\Omega_r) + 2\Omega_r^{3/2}\right)$$

is a solution to the Friedmann equation. Hence calculate a refined estimate for the age and conformal time at recombination using the same parameters as before. [5]