Worksheet: Gaussians, Transfer Functions and Power Spectra

- Fluctuations $\{\Delta_i\}$ are a set of N independent Gaussian random variables with zero mean, each with unknown variance C so that for each one
$$P(\Delta_i|C) = \frac{1}{\sqrt{2\pi C}} e^{-\frac{\Delta_i^2}{2C}}.$$
Show that $P(\{\Delta_i\}|C)$ depends on the 'data' $\{\Delta_i\}$ via the combination $\hat{C} \equiv \frac{1}{N}\sum_i \Delta_i^2$ (and hence that $\hat{C}$ is a *sufficient statistic*: compressing the data into $\hat{C}$ loses no information in that $P(\hat{C}|C)$ contains the same information about C as does $P(\{\Delta_i\}|C)$.

- Define a matrix $\boldsymbol{C}$ with $\boldsymbol{C} = \mathrm{diag}(C, C, C...)$ and combine the $\{\Delta_i\}$ into a vector $\mathbf{a}$. Show that $\mathbf{a}$ has the multi-variate Gaussian distribution
$$P(\mathbf{a}|\boldsymbol{C}) = \frac{1}{(2\pi)^{N/2}|\boldsymbol{C}|^{1/2}} e^{-\mathbf{a}^T\boldsymbol{C}^{-1}\mathbf{a}/2}.$$

- A general N-dimensional Gaussian distribution has the same form, where in general $\boldsymbol{C}$ is not diagonal and encodes the correlation between variables.

  For some *linear* transformation on $\mathbf{a}$ we can write $\mathbf{a}' = \boldsymbol{A}\mathbf{a}$ for a matrix $\boldsymbol{A}$. Show that if $\mathbf{a}$ has a Gaussian distribution with covariance $\boldsymbol{C}$ then $\mathbf{a}'$ also has a Gaussian distribution and find its covariance.

- If a system evolves linearly in time we can write $\mathbf{a}(t) = \boldsymbol{T}(t)\mathbf{a}(0)$. If $\boldsymbol{C}(0)$ is the covariance of $\mathbf{a}(0)$ what is the covariance of $\mathbf{a}(t)$? If $\boldsymbol{C}(0)$ and $\boldsymbol{T}(t)$ are isotropic ($\propto \boldsymbol{I}$) what is $P \equiv \langle [a_i(t)]^2 \rangle$?

- Consider a wave $u(x,t)$ on a long piece of string, satisfying the wave equation
$$\frac{\partial^2 u}{\partial t^2} - c_s^2 \frac{\partial^2 u}{\partial x^2} = 0.$$
The string starts with some initial displacement $u(x,0)$ with $\partial_t u(x,0) = 0$. Fourier transform $u(x,t)$ into $u(k,t)$ and give the equation for $u(k,t)$. Hence show that
$$u(k,t) = \cos(c_s k t) u(k,0).$$

- The statistics of the initial wave are independent of position (homogeneous) so that $\langle u(x,0)u(x+c,0)\rangle$ is independent of x. Show that for this to be true we need
$$\langle u(k,0)u(k',0)\rangle \propto \delta(k+k').$$

- Define the power spectrum $P(k,t)$ by $\langle u(k,t)u(k',t)\rangle = P(k,t)\delta(k+k')$. If $u(k,0)$ are statistically homogeneous and Gaussian with power spectrum $P(k,0)$, what is $P(k,t)$ and what is its distribution?

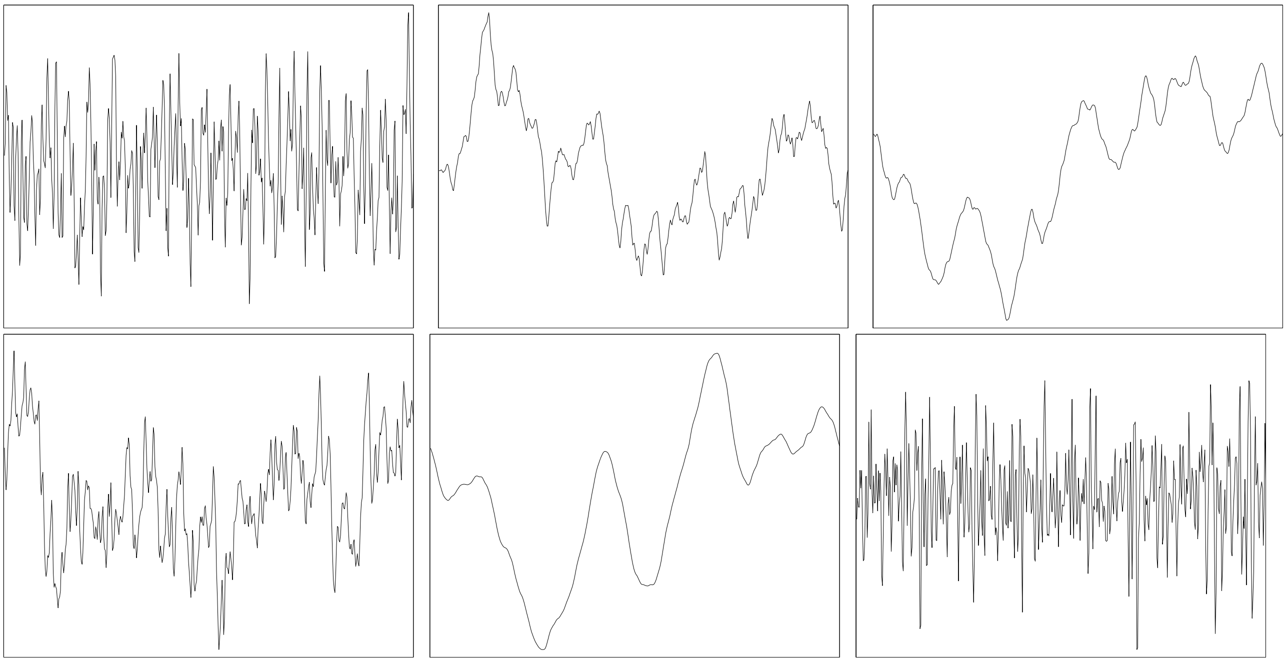
- If $P(k,0) = \text{const}$ over some range of k around k_0, sketch $P(k,t_*)$ at t_* for $t_* \gg 1/(c_s k_0)$, $t_* \sim 1/(c_s k_0)$ and $t_* \ll 1/(c_s k_0)$.

- It the initial displacements $u(x,0)$ are uncorrelated, $\langle u(x,0)u(y,0)\rangle \propto \delta(x-y)$ (*white noise*), and $P(k) \propto k^\alpha$, what value of α would you get?

- To have $\langle [u(x,0)]^2\rangle$ having equal contributions from each $\ln k$ (*scale invariant*) what α would you need?

Realizations of initial conditions for $P(k,0) \propto k^{\alpha}$ for integer α

Which is which?



What would they look like some time later?