Early Universe : 2015 Open Note Test

Answer TWO questions out of three, each of which is marked out of 20.
Time allocated: 1Hr 30 minutes.

1. The effective number degrees of freedom for particles in thermal equilibrium at the same temperature T is

$$g_* = \sum_{i,\text{bosons}} g_i + \frac{7}{8} \sum_{i,\text{fermions}} g_i \, .$$

where g_i are the degeneracy factors. The entropy density is given in terms of the temperature by $s = \frac{2\pi^2}{45} g_* T^3$. Consider the case of (spin 2) gravitons, which decouple at $T \approx 10^{19}$GeV. Assume a hot bing bang model without inflation, so that at $T > 10^{19}$GeV the gravitons and other particles were in thermal equilibrium.

   (a) Describe in your own words what it means for a particle to decouple, and qualitatively what determines when this happens in the early universe. [2]

   (b) Today, the remaining relativistic species (apart from gravitons) are photons and neutrinos, which are decoupled and have different temperatures. What is the total entropy density of photons and neutrinos today in terms of the CMB temperature $T_{\gamma,0}$? [5]

   (c) Assume there are only gravitons and standard model degrees of freedom at $T > 1$TeV, so that the standard model gives $g_* = 106.75$ and all the standard model particles are thermalized at the same temperature. Estimate the temperature T_g today of the gravitons. [7]

   (d) How would T_g change if instead there was a thermalized GUT model with $g_* \approx 1000$ at $T > 10^{16}$GeV? [2]

   (e) Approximately what fraction of the energy density at BBN is in gravitons in the two cases considered above? (assume BBN is after electron-positron annihilation) [4]

2. Assume single-field slow-roll inflation with slow-roll parameters defined by

$$\epsilon_V \equiv \frac{M_P^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2 \qquad \eta_V \equiv M_P^2 \frac{V_{,\phi\phi}}{V} .$$

Also assume results to first order in slow roll, so that the observed scalar spectral index n_s, and tensor amplitude ratio r are given by

$$n_s - 1 = -6\epsilon_V + 2\eta_V \qquad r = 16\epsilon_V .$$

   (a) What is meant by the number of e-foldings N since observable perturbation modes left the horizon, and why do we think N is likely to be in the range $50 \lesssim N \lesssim 60$? [3]

   (b) At any given time during inflation, what is $dN/d\phi$ in terms of the potential $V(\phi)$ and ϕ? We can think of n_s and r as functions of N: show that

$$n_s - 1 = \frac{d \ln r}{dN} - \frac{r}{B},$$

   where B is a constant that you should determine. [8]

   (c) Show that if

$$r = \frac{B}{N + AN^2}$$

   where A is any unknown constant, then $n_s = 1 - 2/N$. [3]

(d) What power law potential would give $r = B/N$ (i.e. $A = 0$) to leading order in $1/N$? Calculate N for a potential of the form $V(\phi) = V_0(1 - e^{-\phi/M_P})$, and show that for $\phi \gg M_P$ during inflation, this potential correspond to the alternative limit in which approximately $r \propto 1/N^2$. [6]

3. Consider a universe filled with a single non-relativistic matter fluid with equation of state $P = \alpha\rho^{4/3}$, where $P \ll \rho$ so that the background evolution is well-approximated as matter domination. Linear perturbations $\Delta \equiv \delta\rho/\rho$ with wavenumber k evolve as

$$\Delta'' + \mathcal{H}\Delta' + \left(k^2 c_s^2 - \frac{3}{2}\mathcal{H}^2\right)\Delta = 0$$

on sub-horizon scales ($k \gg \mathcal{H}$), where $c_s^2 = dP/d\rho$, primes denote derivatives with respect to conformal time, and $\mathcal{H} \equiv a'/a$.

(a) Show that $c_s^2 = c_i^2(\eta_i/\eta)^2$, where c_i and η_i are constants and η is the conformal time. [6]

(b) Find power law solutions of the form $\Delta \propto \eta^\beta$ where the two solutions with different β should be determined. Which solution dominates at late times? [6]

(c) Describe qualitatively why the solution is different on small and large scales (large and small k). At which value of k does the character of the late-time solution change from monotonic to oscillatory? At which value of k does the growing solution become $\Delta =$ constant? [5]

(d) Write the general form of the solution (with two integration constants) at large k where the solution is oscillatory, without any complex numbers so that the solution is manifestly real. Show that the amplitude of the solution decays as $\Delta \propto \eta^{-1/2}$. [3]