Early Universe : 2011 Open Note Test

Answer all questions. Time allocated: 90 minutes. Some integrals that you may find useful are given at the end.

1. It has been suggested that instead of a cosmological constant the dark energy might be due to an exotic fluid with an equation of state $p_{de} = -(2/3)\rho_{de}$ that does not interact with anything else (except via gravity). If a homogenous spatially-flat universe with scale factor a contains only this form of dark energy and pressureless matter with density ρ_m:

   (a) Write down the energy conservation equation for the dark energy fluid, and show that its energy density is proportional to $1/a$. If $\Omega_{de,0} = 0.7$, what was ρ_{de}/ρ_m at redshift 1? [4]

   (b) Give an equation for the Hubble parameter as a function of scale factor in terms of the Hubble parameter today H_0, and the ratios of the energy densities to the critical density today, $\Omega_{m,0}$ and $\Omega_{de,0}$. [1]

   (c) A supernova of known intrinsic luminosity (and no peculiar velocity) is observed at redshift 1. If $\Omega_{de,0} = 3/4$ and H_0 is also measured accurately, what is the expected ratio of the observed flux to the flux that would be observed in a universe where the dark energy is a cosmological constant ($p_{de}/\rho_{de} = -1$)? [5]

2. At least two neutrinos are thought to have a small mass, but small enough that in the early universe the neutrinos are still very relativistic. Assuming zero neutrino chemical potential:

   (a) The equilibrium distribution function at temperature T for a single neutrino species in the limit in which the mass can be neglected, using natural units where $k_B = c = \hbar = 1$, is

   $$f_\nu(p,T) = \frac{g_\nu}{(2\pi)^3}\frac{1}{e^{p/T}+1}$$

   What is the meaning of f_ν and p here, and what is the value of g_ν? [2]

   (b) After a massive neutrino has completely decoupled at temperature T_D and scale factor a_D, show that the energy density in these neutrinos is given by

   $$\rho_\nu = \frac{T_\nu^4}{\pi^2}\int_0^\infty \frac{x^2 dx\sqrt{m_\nu^2/T_\nu^2 + x^2}}{e^x+1}$$

   where $T_\nu \equiv T_D a_D/a$ [assume the neutrino was highly relativistic when it decoupled, so $m_\nu/T_D \ll 1$ is negligible]. [3]

   (c) By considering a series expansion for small m_ν/T_ν show that if there is a massless neutrino with energy density $\rho_{\nu 0}$, for a nearly-relativistic massive neutrino with mass m_ν

   $$\rho_\nu \approx \rho_{\nu 0}\left(1 + \frac{5}{7\pi^2}\frac{m_\nu^2}{T_\nu^2}\right)$$

   to leading order in m_ν/T_ν. You can assume all the neutrinos were in thermal equilibrium before they decoupled. [3]

   (d) The effect of massive neutrinos can be seen in the linear CMB anisotropies if they are massive enough to affect the background evolution before recombination, e.g. when ρ_ν is significantly larger than $\rho_{\nu 0}$. Approximately what is the lightest neutrino (in electron volts) that has an observable effect on the linear CMB anisotropies? [2]

3. Consider a single scalar field model of inflation, with potential given by

$$V(\phi) = m|\phi|^3$$

Restricting to $\phi > 0$ (so that $|\phi| = \phi$):

(a) What are the slow roll parameters in terms of the field value ϕ, and at what values of the field does inflation happen? [2]

(b) What is the number of e-foldings between the beginning and end of inflation if the field starts at the value ϕ_i? Suggest a suitable value of ϕ_i that is sufficient to solve the flatness and horizon problems. [3]

(c) Give the slow roll equations and use them to solve for $\phi(t)$ if $\phi(0) = \phi_i$. [5]

4. Consider the photon fluid in the early universe with background energy density ρ_γ and pressure $p_\gamma = \rho_\gamma/3$, with density perturbation Δ_γ. Before last scattering baryons and photons are tightly coupled, so the photon fluid is tightly-coupled to the baryon fluid with background density ρ_b and negligible pressure $p_b \approx 0$ with density perturbation Δ_b. The tight-coupling means that the fluids cannot flow past one another, so $\mathbf{v}_\gamma = \mathbf{v}_b$. The perturbed Euler equation for the evolution of the joint fluid velocity $\mathbf{v}$ is given (in the conformal Newtonian gauge) by

$$\mathbf{v}' + \mathcal{H}\mathbf{v} + \frac{p'}{\rho + p}\mathbf{v} + \frac{\boldsymbol{\nabla}\delta p}{\rho + p} + \boldsymbol{\nabla}\Psi = 0,$$

where densities and pressures are also for the combined tightly-coupled fluid. Work to linear order in perturbations.

(a) Using the fact that the stress-energy tensor for the composite fluid is conserved and assuming the baryons and photons do not exchange energy, show that

$$\mathbf{v}_\gamma' + \frac{\mathcal{H}R}{1+R}\mathbf{v}_\gamma + \frac{\boldsymbol{\nabla}\Delta_\gamma}{4(1+R)} + \boldsymbol{\nabla}\Psi = 0$$

where $R = 3\rho_b/(4\rho_\gamma)$. Interpret physically the effect of the coupling to the baryons in the terms that are different compared to the form of the equation when there's no baryon coupling (e.g. $R \to 0$). [5]

(b) For sub-horizon perturbations explain why we expect $|\Phi| \approx |\Psi| \ll |\Delta_\gamma|$ during radiation domination. Using this approximation and the result for the evolution of the photon density perturbations

$$\Delta_\gamma' + \frac{4}{3}\boldsymbol{\nabla}\cdot\mathbf{v}_\gamma - 4\Phi' = 0$$

show that the photon density perturbation on sub-horizon scales evolves approximately with

$$\Delta_\gamma'' + \frac{\mathcal{H}R}{1+R}\Delta_\gamma' - \frac{1}{3(1+R)}\boldsymbol{\nabla}^2\Delta_\gamma = \frac{4}{3}\boldsymbol{\nabla}^2\Psi.$$

Compared to the situation with no baryon coupling ($R \to 0$) explain qualitatively the effect of the baryons on the amplitude and wavenumber dependence of the power spectrum of Δ_γ at recombination. [5]

Results you may find useful

$$\int_{1/2}^{1} \frac{dx}{\sqrt{3x^3 + x}} \approx 0.3652 \qquad \int_{1/2}^{1} \frac{dx}{\sqrt{3x^4 + x}} \approx 0.3979$$

$$\int_0^\infty \frac{q\,dq}{e^q + 1} = \frac{\pi^2}{12} \qquad \int_0^\infty \frac{q^3\,dq}{e^q + 1} = \frac{7\pi^4}{120}$$