E/B separation, Pseudo-Cls and sky likelihoods

Antony Lewis[1]
[1] CITA

I. PSEUDO-MULTIPOLES

Observables on the cut sky correspond to a set of Pseudo-Harmonics

$$\tilde{T}_{lm} = \sum_{l'm'} W_{(lm)(lm)'} T_{l'm'} \tag{1}$$

$$\tilde{E}_{lm} = \sum_{l'm'} \left(W_{+(lm)(lm)'} E_{l'm'} + i W_{-(lm)(lm)'} B_{l'm'} \right), \tag{2}$$

$$\tilde{B}_{lm} = \sum_{l'm'} \left(W_{+(lm)(lm)'} B_{l'm'} - i W_{-(lm)(lm)'} E_{l'm'} \right). \tag{3}$$

With the simple case of azimuthal symmetry different m decouple so $W_{(lm)(lm)'} = \delta_{mm'} W_{lml'm}$, etc, (see [1]). The pseudo-Cls are defined in the obvious way

$$\tilde{C}_l^{AB} = \frac{1}{2l+1} \sum_m \tilde{A}^\dagger_{lm} \tilde{B}_{lm}$$

etc, and represent a subset of the available two-point information.

The coupling matrix is given by

$$_sW_{(lm)(l'm')} = \int d\Omega W \, _sY^*_{lm} \, _sY_{l'm'}$$

where W is a positive window function over the sky. Expanding it in harmonics W_{lm} one has

$$_sW_{(l_1m_1)(l_2m_2)} = (-1)^{m_1} \sum_l W_{lm} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \begin{pmatrix} l & l_1 & l_2 \\ 0 & s & -s \end{pmatrix} \begin{pmatrix} l & l_1 & l_2 \\ m & -m_1 & m_2 \end{pmatrix}.$$

Here $m = m_1 - m_2$. Note that in the sum the W_{lm} are only required up to $l_1 + l_2$, the matrix is Hermitian $_sW_{(l_1m_1)(l_2m_2)} = {_sW}^*_{(l_2m_2)(l_1m_1)}$, and $_sW_{(l_1m_1)(l_2m_2)} = (-1)^{m_1+m_2} {_{-s}W}^*_{(l_1-m_1)(l_2-m_2)}$. The polarization coupling matrices are

$$W_{\pm(l_1m_1)(l_2m_2)} = \tfrac{1}{2}\left({_2W}_{(l_1m_1)(l_2m_2)} \pm {_{-2}W}_{(l_1m_1)(l_2m_2)} \right)$$

In the limit $l_{\max} \to \infty$ the $_s\boldsymbol{W}$ are projection operators, so $_s\boldsymbol{W}^2 = {_s\boldsymbol{W}}$. Also for sky window functions W_X and W_Y it follows that

$$_s\boldsymbol{W}^{(X)} {_s\boldsymbol{W}}^{(Y)} = {_s\boldsymbol{W}}^{(XY)} \tag{4}$$

$$\boldsymbol{W}_+^{(X)} \boldsymbol{W}_+^{(Y)} + \boldsymbol{W}_-^{(X)} \boldsymbol{W}_-^{(Y)} = \boldsymbol{W}_+^{(XY)} \tag{5}$$

$$\boldsymbol{W}_+^{(X)} \boldsymbol{W}_-^{(Y)} + \boldsymbol{W}_-^{(X)} \boldsymbol{W}_+^{(Y)} = \boldsymbol{W}_-^{(XY)} \tag{6}$$

where $_s\boldsymbol{W}_{XY}$ is from the product of the two window functions. Hence

$$\begin{pmatrix} \tilde{\mathbf{E}}^{(XY)} \\ \tilde{\mathbf{B}}^{(XY)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{W}_+^{(Y)} & i\boldsymbol{W}_-^{(Y)} \\ \boldsymbol{W}_-^{(Y)} & i\boldsymbol{W}_+^{(Y)} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}}^{(X)} \\ \tilde{\mathbf{B}}^{(X)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{W}_+^{(XY)} & i\boldsymbol{W}_-^{(XY)} \\ -i\boldsymbol{W}_-^{(XY)} & \boldsymbol{W}_+^{(XY)} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}. \qquad (l_{\max} \to \infty)$$

II. E-B SEPARATION

We have the observables

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{W}_+ & i\boldsymbol{W}_- \\ -i\boldsymbol{W}_- & \boldsymbol{W}_+ \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}.$$

To measure the B only we look for a filter matrix $\boldsymbol{P}^\dagger$ such that

$$\boldsymbol{P}_B^\dagger \begin{pmatrix} \boldsymbol{W}_+ \\ -i\boldsymbol{W}_- \end{pmatrix} = 0.$$

(In the azimuthal analysis we effectively used $\boldsymbol{P}_B^\dagger = (0, P_-)$ where P_- projected out the range of $\boldsymbol{W}_-$). We have then have

$$\mathbf{X}_B \equiv \boldsymbol{P}_B^\dagger \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \boldsymbol{P}_B^\dagger \begin{pmatrix} i\boldsymbol{W}_- \\ \boldsymbol{W}_+ \end{pmatrix} \mathbf{B}$$

which have no dependence on $\mathbf{E}$. The projection onto E is generated similarly

$$\boldsymbol{P}_E = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} \boldsymbol{P}_B \qquad \boldsymbol{P}_B = \begin{pmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{pmatrix} \qquad \boldsymbol{P}_E = \begin{pmatrix} \boldsymbol{P}_2 \\ -\boldsymbol{P}_1 \end{pmatrix}.$$

We then have

$$\begin{pmatrix} \mathbf{X}_E \\ \mathbf{X}_B \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_E^\dagger \\ \boldsymbol{P}_B^\dagger \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}\mathbf{E} \\ \boldsymbol{M}\mathbf{B} \end{pmatrix}$$

where

$$\boldsymbol{M} = \boldsymbol{P}_E^\dagger \begin{pmatrix} \boldsymbol{W}_+ \\ -i\boldsymbol{W}_- \end{pmatrix} = \boldsymbol{P}_B^\dagger \begin{pmatrix} i\boldsymbol{W}_- \\ \boldsymbol{W}_+ \end{pmatrix}.$$

A. Method 1

To find $\boldsymbol{P}_B$ we do an SVD

$$\begin{pmatrix} \boldsymbol{W}_+ \\ -i\boldsymbol{W}_- \end{pmatrix} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\dagger.$$

If $\boldsymbol{W}_+$ has dimension $n = (l_{\max} + 1)^2 - 4$, $\boldsymbol{U}$ contains n orthogonal singular vectors. However in general $\boldsymbol{D}$ has some essentially zero elements corresponding to poorly supported modes, so there are effectively only a number $n_D \sim f_{\rm sky} n < n$ supported singular vectors and $\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\dagger \approx \tilde{\boldsymbol{U}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger$, where the tilded variables have the columns and rows corresponding to the nearly zero elements of $\boldsymbol{D}$ removed. Since $\tilde{\boldsymbol{U}}$ then has dimension $2n \times n_D$, it follows that there are $2n - n_D$ other orthogonal vectors making up a $2n \times (2n - n_D)$ matrix $\boldsymbol{P}_B$ such that $\boldsymbol{P}_B^\dagger \tilde{\boldsymbol{U}} = 0 \approx \boldsymbol{P}_B^\dagger \boldsymbol{U}$. Equivalently

$$\boldsymbol{P}_B^\dagger = \boldsymbol{I} - \tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^\dagger$$

where zero rows are deleted. The resultant $\mathbf{X}_B$ then contains $2n - n_D$ elements. The value that counts as 'essentially zero' may be chosen arbitarily small to project out $\mathbf{E}$ to the desired accuracy, and there are always at least n columns in $\boldsymbol{P}_B$ even taking $\tilde{\boldsymbol{D}} = \boldsymbol{D}$. On small patches of sky it is essential to to include the singular vectors corresponding to extremely small values of $\boldsymbol{D}$ as they are poorly determined numerically and may therefore have a non-zero projection onto the eigenvectors required to make up $\boldsymbol{P}_B$.

In general $\boldsymbol{M}$ has many very nearly zero eigenvalues, and one can again use an SVD $\boldsymbol{M} = \boldsymbol{U}_M \boldsymbol{D}_M \boldsymbol{V}_M^\dagger \approx \tilde{\boldsymbol{U}}_M \tilde{\boldsymbol{D}}_M \tilde{\boldsymbol{V}}_M^\dagger$ to find the well determined modes $\tilde{\mathbf{X}}_B \equiv \tilde{\boldsymbol{U}}_M^\dagger \mathbf{X}_B$, where $\dim(\tilde{\mathbf{X}}) \le n$. Note that $\boldsymbol{P}_B^\dagger \boldsymbol{P}_E \neq 0$ in general.

B. E/B separation for band-limited skies

Here we assume there exists some $l_{\max}$ for which all B, *and* E, is negligible for $\ell > l_{\max}$. We do an SVD

$$\begin{pmatrix} i\boldsymbol{W}_- \\ \boldsymbol{W}_+ \end{pmatrix} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\dagger \approx \tilde{\boldsymbol{U}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \quad (7)$$

$$\begin{pmatrix} \boldsymbol{W}_+ \\ -i\boldsymbol{W}_- \end{pmatrix} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\dagger \approx \boldsymbol{R}\tilde{\boldsymbol{U}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \quad (8)$$

where

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{pmatrix}.$$

We then have

$$\tilde{\boldsymbol{U}}^\dagger \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \mathbf{B} + i\boldsymbol{H}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \mathbf{E} \quad (9)$$

$$\tilde{\boldsymbol{U}}^\dagger \boldsymbol{R}^\dagger \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \mathbf{E} - i\boldsymbol{H}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \mathbf{B} \quad (10)$$

where the Hermitian matrix $\boldsymbol{H}$ is defined by

$$i\boldsymbol{H} \equiv \tilde{\boldsymbol{U}}^\dagger \boldsymbol{R}\tilde{\boldsymbol{U}}.$$

The vector $\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}\mathbf{B}$ contains essentially all the observable information about $\mathbf{B}$. Solving we have

$$\tilde{\boldsymbol{U}}^\dagger \left[\boldsymbol{I} - \boldsymbol{R}\tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^\dagger \boldsymbol{R}^\dagger\right] \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = (1 - \boldsymbol{H}^2)\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \mathbf{B} \quad (11)$$

$$\tilde{\boldsymbol{U}}^\dagger \boldsymbol{R}^\dagger \left[\boldsymbol{I} - \tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^\dagger\right] \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{B}} \end{pmatrix} = (1 - \boldsymbol{H}^2)\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \mathbf{E} \quad (12)$$

Thus this is essentially the same as method 1. For band limited skies in the absense of noise there is no information loss. However for modes corresponding to non-zero eigenvalues of $\boldsymbol{H}$ the signal to noise is decreased relative to doing no separation. This is consistent with the understanding that the noise on the second derivatives of the data becomes larger as you near the boundary of the region.

There is no need to actually compute the SVD. We can use

$$\boldsymbol{W}_+^2 + \boldsymbol{W}_-^2 \approx \tilde{\boldsymbol{V}}\tilde{\boldsymbol{D}}^2\tilde{\boldsymbol{V}}^\dagger \quad (13)$$

$$\boldsymbol{W}_+\boldsymbol{W}_- + \boldsymbol{W}_-\boldsymbol{W}_+ \approx -\tilde{\boldsymbol{V}}\tilde{\boldsymbol{D}}\boldsymbol{H}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{V}}^\dagger \quad (14)$$

(approximations only arising from the usual use of tilded variables). This is significantly faster as it only requires diagonalization of one $n \times n$ Hermitian matrix, rather than an SVD of a $2n \times n$ matrix.

The symmetriy of ${}_s\boldsymbol{W}$ implies that $\boldsymbol{P}\boldsymbol{W}_-\boldsymbol{P} = -\boldsymbol{W}_-^*$ for the matrix $P_{(lm)(l'm')} \equiv (-1)^m \delta_{ll'}\delta_{-mm'}$ which satisfies $\boldsymbol{P}^2 = 1$. This implies that eigenvalues of $\boldsymbol{W}_-$ come in $\pm$ pairs:

$$\boldsymbol{W}_-\mathbf{e} = \lambda\mathbf{e} \quad \Longrightarrow \quad \boldsymbol{W}_-\boldsymbol{P}\mathbf{e}^* = -\lambda\boldsymbol{P}\mathbf{e}^*.$$

Hence $\mathrm{Tr}(\boldsymbol{W}_-) = 0$.

Non-band limited skies

.. eigenstructure stuff etc in final paper.

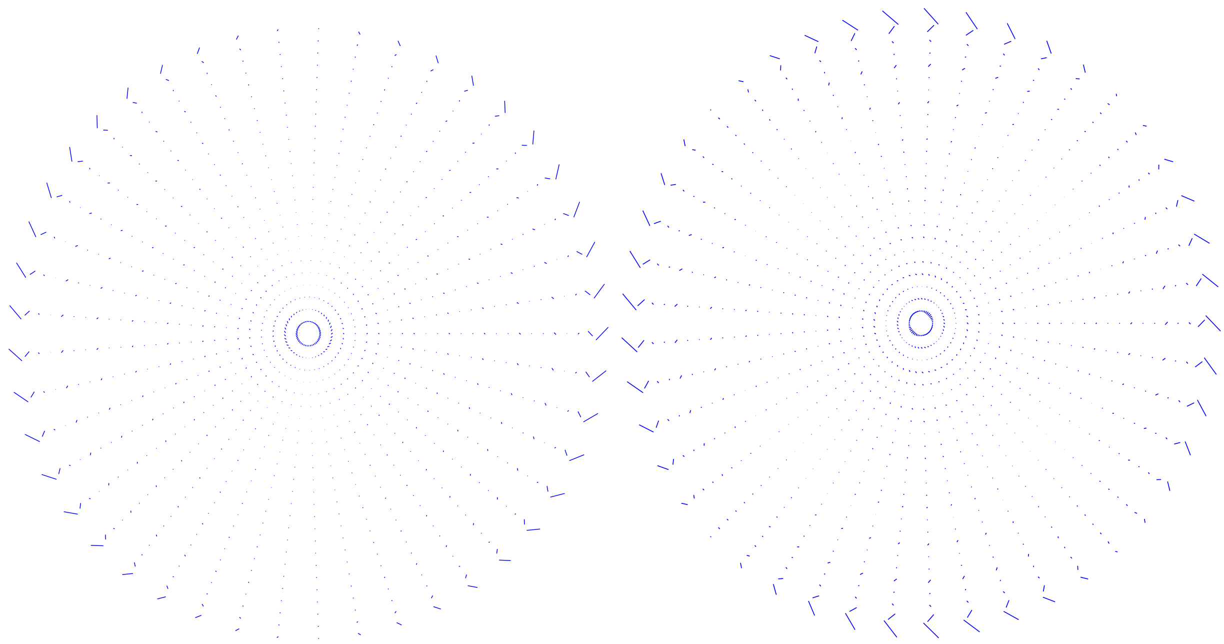

FIG. 1: $m = 1$ and $m = 2$ azimuthal pure B boundary modes othogonal to the modes allowed by the range of $\boldsymbol{W}_-$, for band limited E with $l_{\max} = 300$, $\Theta = 20°$.

C. More

The noise stucture of these $\mathbf{X}$ variables is more complicated than in the original azimuthal prescription. For uncorrelated isotropic noise

$$\left\langle \begin{pmatrix} \mathbf{X}_E \\ \mathbf{X}_B \end{pmatrix} \begin{pmatrix} \mathbf{X}_E^\dagger & \mathbf{X}_B^\dagger \end{pmatrix} \right\rangle_N = \sigma_N^2 \begin{pmatrix} \boldsymbol{P}_E^\dagger \\ \boldsymbol{P}_B^\dagger \end{pmatrix} \begin{pmatrix} \boldsymbol{W}_+ & i\boldsymbol{W}_- \\ -i\boldsymbol{W}_- & \boldsymbol{W}_+ \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_E & \boldsymbol{P}_B \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} \boldsymbol{P}_2^\dagger \boldsymbol{W}_+ \boldsymbol{P}_2 - \boldsymbol{P}_1^\dagger \boldsymbol{W}_+ \boldsymbol{P}_1 & i\boldsymbol{P}_1^\dagger \boldsymbol{W}_- \boldsymbol{P}_1 - i\boldsymbol{P}_2^\dagger \boldsymbol{W}_- \boldsymbol{P}_2 \\ -i\boldsymbol{P}_1^\dagger \boldsymbol{W}_- \boldsymbol{P}_1 + i\boldsymbol{P}_2^\dagger \boldsymbol{W}_- \boldsymbol{P}_2 & \boldsymbol{P}_2^\dagger \boldsymbol{W}_+ \boldsymbol{P}_2 - \boldsymbol{P}_1^\dagger \boldsymbol{W}_+ \boldsymbol{P}_1 \end{pmatrix}. \tag{16}$$

In general $\langle \mathbf{X}_E \mathbf{X}_B^\dagger \rangle_N \neq 0$. One finds that in the azimuthal case the (small number of) modes gained by this method have small non-zero E/B noise correlation.

The azimuthally asymmetric code can be tested easily by rotating an a asimuthally symmetric patch on the sky and checking that $\boldsymbol{W}_-$ has the required number of zero eigenvalues, etc. The method would also work with a set of point samples (where W is a series of delta functions), and so is of possible applicability to weak lensing data in which the samples invariably have a very complicated topology.

D. Azimuthal case

For a circular boundary at constant latitude $\theta = \Theta$ (i.e. the boundary of an azimuthal patch), we find

$$W_{-(lm)(l'm)} = -\frac{m}{2|m|} \left[u_l(m) u_{l'}^*(m) + v_l(m) v_{l'}^*(m) \right]_\Theta, \tag{17}$$

where the vectors

$$u_l(m) = \sqrt{\frac{(l-2)!}{(l+2)!}} \sqrt{8|m|\pi} \sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{Y_{lm}}{\sin\theta} \right) \tag{18}$$

$$v_l(m) = \sqrt{\frac{(l-2)!}{(l+2)!}} \frac{\sqrt{8|m|\pi(m^2-1)}}{\sin\theta} Y_{lm} \tag{19}$$

for $l \geq 2$ and some arbitrary Φ. In the $l_{\max} \to \infty$ limit this is the spectral decomposition, and as expected the eigenvalues are $\pm\frac{1}{2}$. To get the normalization for the vectors we used the results:

$$\sum_{l=|m|}^{\infty} \frac{(l-2)!}{(l+2)!}|Y_{lm}|^2 = \frac{\sin^2\theta}{8|m|\pi(m^2-1)} \qquad (|m| \geq 2) \tag{20}$$

$$\sum_{l=|m|}^{\infty} \frac{(l-2)!}{(l+2)!}Y_{lm}^* \sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{Y_{lm}}{\sin\theta}\right) = 0 \qquad (|m| \geq 2) \tag{21}$$

$$\sum_{l=\max(2,|m|)}^{\infty} \frac{(l-2)!}{(l+2)!}\left|\sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{Y_{lm}}{\sin\theta}\right)\right|^2 = \frac{1}{8|m|\pi} \qquad (|m| \geq 1) \tag{22}$$

I got these from writing the general sum with different arguments as $(\nabla^2+2)\nabla^2$ of the known sum

$$\sum_{l=\max(|s|,|m|)}^{\infty} {}_s\lambda_{lm}(x)\,{}_s\lambda_{lm}(y) = \frac{1}{2\pi}\delta(x-y),$$

then solved explicitly for the Greens' function for the fourth order equation. The normalization of u_l for $m = \pm 1$ follows from knowing the eigenvalues must be $\pm\frac{1}{2}$.

The azimuthal case has the nice property that even for finite $l_{\max}$ one can project out exactly the E contribution on all scales. This is because it is possible to construct scalar window functions zero and with zero derivative on the boundary using only a finite number of harmonics. The matrix $\boldsymbol{W}_-$ trucated at a finite number of rows (but retaining an unlimited number of columns) can be written as

$$\boldsymbol{W}_- = \boldsymbol{U}_-\boldsymbol{D}_-\boldsymbol{V}_-^\dagger$$

where $\boldsymbol{U}_-$ is a column unitary matrix with two columns, and hence the range of $\boldsymbol{W}_-$ can be projected out using

$$\boldsymbol{P}_- \equiv \boldsymbol{I} - \boldsymbol{U}_-\boldsymbol{U}_-^\dagger \qquad \boldsymbol{P}_-\boldsymbol{W}_- = 0 \tag{23}$$

so that

$$\boldsymbol{P}_-\tilde{\mathbf{E}} = \boldsymbol{P}_-\boldsymbol{W}_+\mathbf{E} \qquad \boldsymbol{P}_-\tilde{\mathbf{B}} = \boldsymbol{P}_-\boldsymbol{W}_+\mathbf{B} \tag{24}$$

loosing at most two modes per m (per boundary). The matrix $\boldsymbol{U}_-$ can in practice be constructed from the SVD of the $[l_{\max} - \max(2,|m|)+1] \times 2$ matrix consisting of the two vectors $u_l(m)$ and $v_l(m)$ (the SVD is neccessary because for finite $l_{\max}$ $u_l(m)$ and $v_l(m)$ are no longer normalized or orthogonal). Thus two maps consisting of pure E and pure B can be constructed simply from

$$\tilde{\mathbf{E}}' = (\boldsymbol{I} - \boldsymbol{U}_-\boldsymbol{U}_-^\dagger)\tilde{\mathbf{E}} \qquad \tilde{\mathbf{B}}' = 0 \qquad \text{(pure E)} \tag{25}$$

$$\tilde{\mathbf{B}}' = (\boldsymbol{I} - \boldsymbol{U}_-\boldsymbol{U}_-^\dagger)\tilde{\mathbf{B}} \qquad \tilde{\mathbf{E}}' = 0 \qquad \text{(pure B).} \tag{26}$$

If desired a map of the ambiguous modes can also be constructed from the remaining modes.

E. Junk

$$\sum_{l=2}^{\infty} \frac{(l-2)!}{(l+2)!}Y_{lm}^*\partial_\theta Y_{lm} = \frac{\sin(2\theta)}{16|m|\pi(m^2-1)} \tag{27}$$

$$\sum_{l=2}^{\infty} \frac{(l-2)!}{(l+2)!}|\partial_\theta Y_{lm}|^2 = \frac{m^2-\sin^2\theta}{8|m|\pi(m^2-1)} \tag{28}$$

III. CUT-SKY TEMPERATURE ANALYSIS

Averaging over realisations

$$\langle \tilde{C}_l^{TT} \rangle = M_{ll'} B_{l'}^2 C_{l'}^{TT}$$

where the coupling matrix is given by

$$M_{ll'} = \frac{1}{2l+1} \sum_{m,m'} |W_{lml'm'}|^2 = \frac{(2l'+1)}{4\pi} \sum_{l''} W_{l''} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2$$

where $W_l = \sum_m |W_{lm}|^2$. An unbiassed estimator fo the C_l is given by $\boldsymbol{M}^{-1}\tilde{\mathbf{C}}$. The WMAP approximation for the Fisher matrix in the zero noise limit is given by

$$F_{ll'}^{WMAP} = \frac{2l+1}{2C_l C_l'} M_{ll'}$$

and they note that $M_{ll} \sim (2l+1) f_{sky}^2$. One can remove possible $l = 0, 1$ contamination by using $\tilde{\mathbf{T}}' \equiv \boldsymbol{P}\tilde{\mathbf{T}}$, where $\boldsymbol{PW}$ only couples $l = 0, 1$ into the first few elements of $\tilde{\mathbf{T}}'$. For large patches of sky $\boldsymbol{P}$ is close to the identity and does not disturb the l, m ordering too much.

To compute the likelihood essentially exactly we use a simple trivialisation of the polarization analysis in [1] (see [2, 3]). We use the cut sky observables $\tilde{\mathbf{T}}' \equiv \boldsymbol{PW}\mathbf{T}$. The rectangular matrix P is used to project out the $l = 0, 1$ modes, and is constructed using partial Householder transformations [3]. P can also reduce the size of $\tilde{\mathbf{T}}'$ by including only those modes that contain all the information below some $l'_{\max}$. Use SVD to write $\boldsymbol{PW} = \boldsymbol{U}\bar{\boldsymbol{D}}\boldsymbol{V}^\dagger$ where $\bar{\boldsymbol{D}}$ is diagonal and $\boldsymbol{V}$ is column unitary we have the vector of observables

$$\mathbf{X} \equiv \boldsymbol{D}^{-1/2}\boldsymbol{U}^\dagger \boldsymbol{P}\tilde{\mathbf{T}} = \boldsymbol{D}^{1/2}\boldsymbol{V}^\dagger \mathbf{T}.$$

Here $\boldsymbol{D}$ is obtained from $\bar{\boldsymbol{D}}$ by deleting rows with small ($\ll 1$) diagonal elements corresponding to the poorly supported eigenfunctions. Alternatively one could apply the projection after establishing the well determined modes.

The vector $\mathbf{X}$ then contains essentially all of the cosmological information on the cut sky, and, since they are linear combinations of (assumed) Gaussian variables, they are also Gaussian so the likelihood can be calculated from

$$L(\mathbf{X}|\mathbf{C}) \propto \frac{\exp[-\frac{1}{2}\mathbf{X}^\dagger(\boldsymbol{N}+\boldsymbol{S})^{-1}\mathbf{X}]}{|\boldsymbol{N}+\boldsymbol{S}|^{1/2}}$$

where

$$\boldsymbol{S} \equiv \boldsymbol{D}^{1/2}\boldsymbol{V}^\dagger \mathrm{diag}(\mathbf{C}) \boldsymbol{V}\boldsymbol{D}^{1/2}$$

and $\boldsymbol{N}$ is a noise matrix. For $l \lesssim 30$ this can be evalulated in the order of seconds.

As for the WMAP approximation, it doesn't seem to bias results significantly. The likelihood for C_2 is maybe off by a factor of three for ΛCDM models, but for models close by the slope is not far off so the likelihood ratio is about right, and it doesn't appear to give any significant bias on n_{run}.

IV. POLARIZATION PSEUDO-CLS

Here we consider azimuthal patches. Writing $W_{ll'}^m \equiv W_{lml'm}$ they have expectation values (over realizations) of the form

$$\langle \tilde{C}_l^{EE} \rangle = X_{+,ll'} B_{l'}^2 C_{l'}^{EE} + X_{-,ll'} B_{l'}^2 C_{l'}^{BB} + \tilde{N}$$

etc, where the coupling matrices are

$$X_{+,ll'} = \frac{1}{2l+1} \sum_m W_{+ll'}^m{}^2 \qquad X_{-,ll'} = \frac{1}{2l+1} \sum_m W_{-ll'}^m{}^2$$

N is the non band-limited noise and B_l^2 results from convolution with the beam. The covariances are

$$\langle \Delta\tilde{C}_l^{TT}\Delta\tilde{C}_{l'}^{TT}\rangle = \frac{2}{(2l+1)(2l'+1)}\sum_m |\sum_{l_1} W^m_{ll_1}W^m_{l'l_1}C^{(TT)}_{l_1}|^2 \tag{29}$$

$$\langle \Delta\tilde{C}_l^{EE}\Delta\tilde{C}_{l'}^{EE}\rangle = \frac{2}{(2l+1)(2l'+1)}\sum_m \left(|\sum_{l_1} W^m_{+ll_1}W^m_{+l'l_1}C^{(EE)}_{l_1}|^2 + |\sum_{l_1} W^m_{-ll_1}W^m_{-l'l_1}C^{(BB)}_{l_1}|^2\right) \tag{30}$$

$$\langle \Delta\tilde{C}_l^{BB}\Delta\tilde{C}_{l'}^{EE}\rangle = \frac{2}{(2l+1)(2l'+1)}\sum_m \left(|\sum_{l_1} W^m_{+ll_1}W^m_{-l'l_1}C^{(BB)}_{l_1}|^2 + |\sum_{l_1} W^m_{-ll_1}W^m_{+l'l_1}C^{(EE)}_{l_1}|^2\right) \tag{31}$$

$$\langle \Delta\tilde{C}_l^{TE}\Delta\tilde{C}_{l'}^{TE}\rangle = \frac{1}{(2l+1)(2l'+1)}\sum_m \Bigg(\sum_{l_1} W^m_{ll_1}W^m_{+l'l_1}C^{(TE)}_{l_1}\sum_{l_2} W^m_{+ll_2}W^m_{l'l_2}C^{(TE)}_{l_2} \tag{32}$$

$$+\sum_{l_1} W^m_{ll_1}W^m_{l'l_1}C^{(TT)}_{l_1}\left\{\sum_{l_2} W^m_{+ll_2}W^m_{+l'l_2}C^{(EE)}_{l_2} + W^m_{-ll_2}W^m_{-l'l_2}C^{(BB)}_{l_2}\right\}\Bigg) \tag{33}$$

$$\langle \Delta\tilde{C}_l^{TT}\Delta\tilde{C}_{l'}^{TE}\rangle = \frac{2}{(2l+1)(2l'+1)}\sum_m \left(\sum_{l_1} W^m_{ll_1}W^m_{l'l_1}C^{(TT)}_{l_1}\right)\left(\sum_{l_2} W^m_{ll_2}W^m_{+l'l_2}C^{(TE)}_{l_2}\right) \tag{34}$$

..and so on for the other ones (for full-sky analysis see [4]). This all generalized straightforwardly to the case when the coupling matrices include a non-isotropic azimuthally symmetric noise variance. The enthusiast could also compute the skewness of the estimators analytically (e.g. see [5]) and fit to an offset-lognormal, etc... Including C^{TB} etc does not gain anything since the covariance is diagonal: they have non-zero variance, but uncorrelated with the ones of interest.

In the case of equatorial sky cut with $b < 45^\circ$ the coupling matrix X_+ is well conditioned and one can solve for the polarization C_l directly

$$C^{(EE)} = (X_+ - X_- X_+^{-1} X_-)^{-1}\left\langle \tilde{C}^{EE} - X_- X_+^{-1}\tilde{C}^{BB}\right\rangle.$$

This is presumably equivalent to the SPICE estimator when the correlation functions are available over the full angular range. When less than this range is available (e.g. $b > 45^\circ$) X_+ becomes badly conditioned and one needs to introduce binning, in the same way that the SPICE estimators come to depend on the C_l over a particular window.

[1] A. Lewis, A. Challinor, and N. Turok, Phys. Rev. D **65**, 023505 (2002), astro-ph/0106536.
[2] K. M. Górski, Astrophys. J. Lett. **430**, 85 (1994), astro-ph/9403066.
[3] D. J. Mortlock, A. D. Challinor, and M. P. Hobson (2000), astro-ph/0008083.
[4] M. Kamionkowski, A. Kosowsky, and A. Stebbins, Phys. Rev. D **55**, 7368 (1997), astro-ph/9611125.
[5] B. D. Wandelt, E. Hivon, and K. M. Górski, Phys. Rev. D **64**, 083003 (2001), astro-ph/0008111.