Galaxy lensing shape estimation notes

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I. ELLIPTICITY ESTIMATORS

Start off with perfect resolution zero-noise case.

Under lensing the observed (zero-PSF) intensity $I(\mathbf{x}) \to I(S\mathbf{x})$. In general a unique positive definite symmetric solution $\boldsymbol{M}$ to

$$\int d^2\mathbf{x} f(I(\mathbf{x}), \boldsymbol{M}^{-1/2}\mathbf{x}) = 0$$

for some function f will transform under lensing as $\boldsymbol{M} \to S^{-1}\boldsymbol{M}S^{-1}$, as is readily shown by a change of variables. Hence $\boldsymbol{M}$ transforms like a quadrupole moment, and can be used to define a valid ellipticity estimator regardless of the distribution of $I(\mathbf{x})$. Definitions of $\boldsymbol{M}$ in this class include least squares fits to the data that minimize

$$\int d^2\mathbf{x} |I(\mathbf{x}) - g(\boldsymbol{M}^{-1/2}\mathbf{x})|^2.$$

It also includes maximum likelihood fits of a model g that minimize

$$\chi^2 = \int d^2\mathbf{x} \left(I(\mathbf{x}) - g(\boldsymbol{M}^{-1/2}\mathbf{x})\right)^T [N(I(\mathbf{x}), \boldsymbol{M}^{-1/2}\mathbf{x})]^{-1} \left(I(\mathbf{x}) - g(\boldsymbol{M}^{-1/2}\mathbf{x})\right).$$

Here the 'noise' N can vary spatially depending on $I(\mathbf{x})$ or $\boldsymbol{M}^{-1/2}\mathbf{x}$, for example Poisson + sky noise, but cannot vary with any other function of $\mathbf{x}$. Thus model-fittings of the form $g(\boldsymbol{M}^{-1/2}\mathbf{x})$ will define valid ellipticity estimators regardless of whether or not the model is at all like $I(\mathbf{x})$.

A further class of valid estimators are the adapted weighted quadrupole moments

$$\hat{\boldsymbol{M}} \propto \int d^2\mathbf{x} q(I(\mathbf{x}))\mathbf{x}\mathbf{x}^T f(\boldsymbol{M}^{-1/2}\mathbf{x})$$

where $\boldsymbol{M}$ is defined via a maximum-likelihood or least-squares fit, and f is some window function adapted to the shape of $I(\mathbf{x})$ via the dependence on $\boldsymbol{M}^{-1/2}\mathbf{x}$, and q is any function. For $f = \text{const}$ and $q = I$ these are just the unweighted quadrupole moments, which unfortunately have divergent noise properties. However the weighted moments can have well defined noise properties and make a useful estimator. One particular case of potential interest is

$$\hat{\boldsymbol{M}} \propto \int d^2\mathbf{x} I(\mathbf{x})\mathbf{x}\mathbf{x}^T \frac{g(\boldsymbol{M}^{-1/2}\mathbf{x})}{\sigma_N^n + g(\boldsymbol{M}^{-1/2}\mathbf{x})}$$

which is the unweighted moment for signal dominated sources, and the adapted weighted moment in the noise dominated limit. However the window function depends on the magnitude of the source.

A. Multiple fits

We may also fit functions of several different $\boldsymbol{M}_i$:

$$\int d^2\mathbf{x} f(I(\mathbf{x}), \{\boldsymbol{M}_i^{-1/2}\mathbf{x}\}) = 0.$$

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*URL: http://cosmologist.info

If the $\boldsymbol{M}_i$ are distinct, any linear combination may be used to define a quantity that transforms as a quadrupole moment and hence forms a valid ellipticity estimator. For multiple linear fits of the same function minimizing

$$\int d^2\mathbf{x}|I(\mathbf{x}) - \sum_i A_i g(\boldsymbol{M}_i^{-1/2}\mathbf{x})|^2$$

there are multiple degenerate solutions, so the quadrupole estimator should inherit the symmetries of the fit. For a sum of non-aligned Gaussians this includes the weighted quadrupole moment of the model:

$$\hat{\boldsymbol{M}} \propto \int d^2\mathbf{x} q(I(\mathbf{x}))\mathbf{x}\mathbf{x}^T \sum_i A_i g(\boldsymbol{M}_i^{-1/2}\mathbf{x})).$$

This includes $q(I(\mathbf{x})) = \text{const}$ corresponding to the quadrupole moments of the model, however this estimator may not be at all optimal with non-zero noise and when the model doesn't fit very well. The optimal solution appears to be unobvious (and in general depends on the distribution of substructure shapes in galaxies).

B. Adaptive moments

Consider the least squares fit [1, 2]

$$\int d^2\mathbf{x}\, |I(\mathbf{x}) - Aw(\alpha)|^2$$

of some aligned elliptical isophote model $w(\alpha)$ where

$$\alpha \equiv (\mathbf{x} - \mathbf{x}_0)^T \boldsymbol{M}^{-1}(\mathbf{x} - \mathbf{x}_0)/2.$$

Minimizing with respect to the parameters gives (assuming $w(r^2 \to \infty) \to 0$ sufficiently rapidly)

$$\boldsymbol{M} = -2\frac{\int d^2\mathbf{x} I(\mathbf{x}) w'(\alpha)(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^T}{\int d^2\mathbf{x} I(\mathbf{x}) w(\alpha)}$$

$$\mathbf{x}_0 = \frac{\int d^2\mathbf{x} I(\mathbf{x}) w'(\alpha)\mathbf{x}}{\int d^2\mathbf{x} I(\mathbf{x}) w'(\alpha)}.$$

$$A = \frac{\int d^2\mathbf{x} I(\mathbf{x}) w(\alpha)}{\int d^2\mathbf{x}[w(\alpha)]^2}$$

where the prime denotes the derivative with respect to α. In the case when the model is correct, $I = Aw$, M is the optimal estimator for a white noise model, with the optimal window being $\propto w'(\alpha) \propto 1/r(dw/dr) \propto (dI/dr)/r$ where $r = \alpha^{1/2}$, in agreement with BJ02 [1, Eq (3.9)]. In general the maximum likelihood M will give the optimal estimator for general noise when w is the correct model. However in general the solution is also non-linear in I and therefore depends on the magnitude of the source.

For uncorrelated Poisson plus sky noise we want to minimize

$$\int d^2\mathbf{x}\, |I(\mathbf{x}) - Aw(\alpha)|^2\, (I(\mathbf{x}) + \sigma_N^2)^{-1}.$$

If w is an exact fit this gives a window function proportional to $w'(\alpha)/(I(\mathbf{x}) + \sigma_N^2)$, so $w \propto (dI/dr)/r(I(\mathbf{x}) + \sigma_N^2)$. When w (and I) is Gaussian, this goes to a constant when $I(\mathbf{x}) \gg \sigma_N^2$, and hence the best fit Gaussian model is measuring the unweighted quadrupole moment. For finite $I(\mathbf{x})$ the window is constant where the signal dominates, but falls off as soon as the noise becomes important. It is therefore very close to what one might expect to be the best quadrupole estimator. However for non-Gaussian galaxies the signal-dominated window it not constant.

The form $w(\alpha) = \exp(-\alpha^n)$ includes Gaussian ($n = 1$), exponential ($n = 1/2$) and top-hat ($n \to \infty$) fits, and has

$$\boldsymbol{M} = 2\frac{\int d^2\mathbf{x} I(\mathbf{x}) e^{-\alpha^n} n\alpha^{n-1}(\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^T}{\int d^2\mathbf{x} I(\mathbf{x}) e^{-\alpha^n}}.$$

Relevant integrals are of the form

$$\int_0^\infty dr r^m e^{-(r^2/2)^n} = 2^{(m-1)/2} n^{-1} \Gamma((m+1)/2n).$$

In general the weighted moment

$$\hat{\boldsymbol{M}} = \frac{2^{1/n}\Gamma(1/n)}{\Gamma(2/n)} \frac{\int d^2\mathbf{x} I(\mathbf{x}) e^{-\alpha^n} (\mathbf{x}-\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T}{\int d^2\mathbf{x} I(\mathbf{x}) e^{-\alpha^n}}$$

differs from M (and will be less optimal for good fits), however in the special case of a Gaussian ($n = 1$) they are the same for any $I(\mathbf{x})$. Thus the least-squares fit Gaussian model has its own adapted weighted quadrupole moments. The constant has been chosen so that they are the same in all cases if the model is exact $I \propto e^{-\alpha^n}$. In general these both differ from the estimator defined as the quadrupole moment of the best-fit model:

$$Q_w = \frac{\int d^2\mathbf{x} e^{-\alpha^n} (\mathbf{x}-\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^T}{\int d^2\mathbf{x} e^{-\alpha^n}} = \frac{\Gamma(2/n)}{\Gamma(1/n)} M.$$

Aside: note trivial proof for the formula for the derivative of a symmetric determinant follows from differentiating both sides of

$$(2\pi)^{n/2} |\boldsymbol{M}|^{-1/2} = \int d^n\mathbf{x} \exp(-\mathbf{x}^T \boldsymbol{M} \mathbf{x}/2)$$

C. PSF

Take the PSF function to be $F(\mathbf{x})$, giving the observed intensity $I_o = F \otimes I$. For general F and any I, the unweighted moments simply add

$$Q_o = Q_F + Q_I.$$

If we fit a model to the galaxies and stars, and the model is exact, this relation can be used to correct for the PSF exactly using the unweighted moments of the models, as long as we don't mind using the potentially noisy unweighted moments. If either of the models is wrong, this won't work.

If we know F we can fit to a PSF-convolved model minimizing,

$$\int d^2\mathbf{x} |I_o(\mathbf{x}) - AF \otimes g(M^{-1/2}\mathbf{x})|^2$$

However this will only give a valid estimator of the shape if *both* F is exactly correct, *and* g is an exact fit to I so that the solution is $I - g = 0$. We can however deconvolve the observation minimizing in harmonic space

$$\int d^2\mathbf{k} \left| \frac{I_o(\mathbf{k})}{F(\mathbf{k})} - A\tilde{g}(M^{1/2}\mathbf{k}) \right|^2$$

which will give a valid estimator conditional only on F being correct. The task then is to regulate the noise properties. Most natural perhaps would be to instead minimize the chi-squared for the deconvolved noise

$$\chi^2 = \left(\frac{I_o(\mathbf{k})}{F(\mathbf{k})} - A\tilde{g}(M^{1/2}\mathbf{k}) \right)^T F(\mathbf{k}) N^{-1}(\mathbf{k}, \mathbf{k}') F(\mathbf{k}') \left(\frac{I_o(\mathbf{k}')}{F(\mathbf{k}')} - A\tilde{g}(M^{1/2}\mathbf{k}') \right)$$

(sum of $\mathbf{k}$'s implicit). However the minimizing M is no longer a valid ellipticity estimator (unless g is an exact model) because of the noise dependence on $\mathbf{k}$ via F, so called 'noise bias'.

One can also fit to an averaged galaxy (averaged over a region of constant shear and PSF), e.g. minimizing

$$\int d^2\mathbf{x} | \sum_i \beta_i I_{o,i}(\mathbf{x}) - AF \otimes g(M^{-1/2}\mathbf{x})|^2$$

This will work conditional only on modelling the PSF and averaged galaxy accurately [3]. Fitting the averaged galaxy profile should be much easier. Kuijken claims that fitting each galaxy separately works almost as well (purely empirically from simulation).

[1] G. M. Bernstein and M. Jarvis, Astrophys. J. **123**, 583 (2002), astro-ph/0107431.
[2] C. M. Hirata and U. Seljak, Mon. Not. Roy. Astron. Soc. **343**, 459 (2003), astro-ph/0301054.
[3] K. Kuijken (1999), astro-ph/9904418.