

CAMB Notes

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Brief summary of some CAMB definitions, parameterizations, and details of internal implementation. Plus some random notes.

I. THE CAMB CODE

The physics is contained in the `equations.f90` file. Everything is in conformal time and units of megaparsecs. The `dtanda` routine returns $1/a'$ (scale factor $a = S$), where the conformal Hubble rate is a'/a . This is used to compute the conformal age of the universe, and background evolution for the pre-computed ionization history and should always be finite. Background density variables include a factor of a^2 , like `grho` $\equiv \kappa a^2 \rho$ where $\kappa \equiv 8\pi G$.

The initial conditions for the scalars and tensors are in the `initial` and `initialt` routines (see above for details). The `derivs` and `derivst` functions return the differential equations - a vector y' of derivatives of the variables being evolved.

The `output` and `outputt` routines calculate the sources for integration against the Bessel functions.

A. Gauge and perturbation definitions

CAMB propagates the covariant equations in the zero-acceleration frame, in which the CDM velocity is zero. This is equivalent to the synchronous gauge.

Note that CAMB uses the variable `etak` $= k\eta_s$, where $\eta_s = -\eta/2$ (in the CDM frame) is the usual synchronous gauge variable. Here η is the curvature perturbation variable as in the previous section. The fractional density perturbation variables are called `clxc` (Δ_c), `clxg` (Δ_γ), `clxr` (Δ_ν), `clxb` (Δ_b) etc. The synchronous gauge variable h_s appears in the CDM frame as $h'_s = 6h' = 2k\mathcal{Z}$. The $l = 1$ moments are handled in terms of the heat fluxes q_i where $\rho_i q_i = (\rho_i + p_i)v_i$. The total heat flux appears as `dgq` $= \kappa a^2 \sum_i \rho_i q_i$ in the code, and the total matter perturbation is `dgrho` $= \kappa a^2 \sum_i \rho_i \Delta_i$.

Other variables are the shear σ and perturbation to the expansion rate Z . They are related by

$$\frac{2}{3}k^2(\beta_2\sigma - Z) = \kappa a^2 \sum_i \rho_i q_i = \text{dgq}$$

$$k^2\eta = \kappa a^2 \sum_i \rho_i \Delta_i - 2k\mathcal{H}Z = \text{dgrho} - 2k\mathcal{H}Z.$$

In terms of synchronous gauge variables $\sigma = (\dot{h}_s + 6\dot{\eta}_s)/2k$. Note that in the non-flat case η_s above is $\beta_2\eta$ of Kodama & Sasaki.

II. NEUTRINOS AND DARK RADIATION

We assume a Fermi-Dirac distribution for neutrinos when they are relativistic. For the massless case the result is independent of the Fermi-Dirac assumption, so massless dark radiation can be modelled as effective neutrinos. When the particles have non-negligible mass, the Fermi-Dirac assumption is not general (it is a good approximation for massive active neutrinos, and is also equivalent via parameter redefinition with the Dodelson-Widrow model for sterile neutrinos (see [1–3]), but may not be correct for more general forms of massive dark radiation).

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Massive neutrinos are assumed to decouple when relativistic and remain exactly thermal with zero chemical potential, so eigenstate i has occupation number (in natural units)

$$\mathcal{N}_i(p) = \frac{1}{e^{\frac{p}{T_i}} + 1} \quad (1)$$

where p is the momentum. All equations for massive neutrinos have mass dependence determined by the dimensionless ratio used in the code `nu_masses(i)` $\equiv m_i c^2 / (k_B T_i^0)$, which determines the particle velocities as a function of redshift. Neutrinos are assumed to be highly non-relativistic today, so

$$\rho_i^0 = m_i n_i^0 = f_i \Omega_\nu \rho_c^0 \quad (2)$$

where f_i is the fraction of the total neutrino energy density in eigenstate i , and ρ_c is the critical density. Number densities are set from the thermal distribution at decoupling while relativistic, where for a Fermion with spin degeneracy two:

$$n_i = \frac{3\zeta(3)}{2\pi^2} \frac{(k_B T_i)^3}{\hbar^3 c^3}. \quad (3)$$

The decoupling of the standard active neutrinos is nearly, but not entirely, complete by the time of electron-positron annihilation. This leads to a slight heating of the neutrinos in addition to that expected for the photons and hence to a small departure from the thermal equilibrium prediction $T_\gamma = (11/4)^{1/3} T_\nu$ between the photon temperature T_γ and the reference neutrino temperature T_ν defined by this relation. The actual temperature T_i for any eigenstate is given by reference to the thermal result T_ν , so that

$$T_i \equiv g_i^{1/4} T_\nu, \quad (4)$$

where this defines g_i , the neutrino degeneracy factor. Hence while relativistic

$$n_i = g_i^{3/4} n_\nu \quad \rho_i = \frac{7}{8} a_R T_i^4 = g_i \rho_\nu = g_i \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} \rho_\gamma, \quad (5)$$

where n_ν and ρ_ν are the results for the thermal prediction. ρ_ν (for one massless neutrino) is in the code as the variable `grhor` $\equiv 8\pi G \rho_\nu^0 / c^2$, which is used to set the densities of the actual species. The dimensionless mass variable is given by

$$\frac{m_i}{T_i^0} = \frac{f_i \Omega_\nu \rho_c^0}{n_i^0 T_i^0} = \frac{f_i \Omega_\nu \rho_c^0}{g_i n_\nu^0 T_\nu^0} \propto \frac{f_i \Omega_\nu \rho_c^0}{g_i \rho_\nu^0}. \quad (6)$$

For each eigenstate the code only needs m_i/T_i^0 and $\rho_i = g_i \rho_\nu$, the latter being stored in `grhormass(i)` $= g_i \times \text{grhor}$ and `grhornomass` for massless neutrinos.

Note that if we fix the total Ω_ν , having N_s neutrino eigenstates with equal g_i and f_i (and hence equal mass), is code-equivalent to having one neutrino with $g = \sum_i g_i$, $f = \sum f_i$, since m_i/T_i (which determines particle velocities) and $\sum_i \rho_i$ (which determines the density) remain the same, even though physically having one state with $g = \sum_i g_i$ corresponds to having a smaller number density of hotter more-massive neutrinos. c_i neutrinos of degenerate mass and the same temperature can therefore modelled simply as a single state with appropriately multiplied $g_i^{\text{eff}} = c_i g_i$.

The physical mass m_i inferred from $\Omega_\nu h^2$ does of course depend on the number of physical neutrino eigenstates. Assuming the neutrinos are highly non-relativistic today, we have

$$\Omega_\nu h^2 \approx \frac{16\zeta(3)}{11\pi} (k_B T_\gamma^0)^3 \frac{G h^2}{c^5 \hbar^3 H_0^2} \sum_i c_i g_i^{3/4} m_i \approx \frac{\sum_i c_i g_i^{3/4} m_i}{94.07 \text{ eV}}. \quad (7)$$

(taking $T_\gamma^0 = 2.7255\text{K}$).

Energy density in neutrinos or dark radiation is conventionally described using an effective energy density number N_{eff} when relativistic, so $N_{\text{eff}} = \sum_i c_i g_i$. The default assumption is three active neutrinos with $N_{\text{eff}} = 3.046$ [4, 5], corresponding to $g_i = 3.046/3$ for each of three neutrinos.

CAMB input parameters

The degeneracy parameters can be specified explicitly for each eigenstate (and the massless neutrino), or the degeneracies can be set internally to give a total N_{eff} assuming all eigenstates have the same temperature, which is the most natural assumption for the case of three active neutrinos. This is determined by the `share_delta_neff` parameter, as described further below. In all cases when there is more than one massive eigenstate `nu_mass_fractions = f1 f2 ...` determines the fraction of the energy density today $\Omega_\nu h^2$ that is associated with each eigenstate, and should sum to 1, e.g. `nu_mass_fractions = 0.75 0.25` to have the first state of two eigenstates accounting for three quarters of the total neutrino density today. When run CAMB prints out the physical eigenstate masses, calculated assuming a thermal distribution (different in the Dodelson-Widrow re-interpretation).

1. `share_delta_neff = T`

In this case all neutrinos are set to have the same g_i per physical mass eigenstate. `massive_neutrinos = c1 c2...` is an array of `nu_mass_eigenstates` integers, specifying the physical number of neutrinos per (degenerate) eigenstate. The non-integer part of the real number `massless_neutrinos` is used to determine the neutrino temperature, so the total effective density parameter is $N_{\text{eff}} = \text{massless_neutrinos} + \sum_i c_i$ and for all species

$$g_i = \frac{N_{\text{eff}}}{[N_{\text{eff}}]} \quad (8)$$

where $[\]$ denotes the integer part.

For example, for the simple case of one massive neutrino and two massless neutrinos, all with standard temperatures so $N_{\text{eff}} = 3.046$, you could use `massless_neutrinos = 2.046`, `massive_neutrinos = 1`.

For an inverted hierarchy of degenerate mass, you could have `massless_neutrinos = 0.046`, `massive_neutrinos = 3`.

For mass splitting but with two degenerate, e.g.: `nu_mass_eigenstates = 2`, `massless_neutrinos = 0.046`, `massive_neutrinos = 2 1`, `nu_mass_fractions = 0.7 0.3`.

Note that since the non-integer part of N_{eff} is used to define the temperature, the physical model for massive neutrinos is discontinuous as the total crosses an integer boundary: $N_{\text{eff}} = 2.99$ is two physical neutrinos with high temperature, $N_{\text{eff}} = 3.046$ is three physical neutrinos with standard temperature. However the results are continuous, only the interpretation in terms of physical mass changes abruptly. Also note that for `massless_neutrinos < 1`, no massless neutrinos are actually included unless `massive_neutrinos = 0`.

2. `share_delta_neff = F`

This allows an arbitrary specification of different neutrino temperatures and masses, where now `nu_mass_degeneracies = g1eff g2eff ...` must be specified explicitly. `massless_neutrinos` determines the g for massless neutrinos. The physical numbers of neutrinos is still specified by `massive_neutrinos`, but does not affect the result except when writing out the physical mass: power spectra etc. from `massive_neutrinos = 2 1` should be the same as from `massive_neutrinos = 1 1` for the same `nu_mass_degeneracies`, but the physical neutrino mass of the first eigenstate is larger in the second case.

Note that the degeneracy inputs are the total, i.e. $g_i^{\text{eff}} = c_i g_i$, so `nu_mass_degeneracies = 2`, `massive_neutrinos = 2` corresponds to two neutrinos with thermal temperature T_ν (contributing 2 to the total N_{eff}), `nu_mass_degeneracies = 1`, `massive_neutrinos = 2` corresponds to two neutrinos with reduced temperature (contributing 1 to the total N_{eff})

3. *Further examples*

To within rounding these combinations of input parameters are equivalent: `share_delta_neff = T`, `massive_neutrinos = 1`, `massless_neutrinos = 2.046` and `share_delta_neff = F`, `massive_neutrinos = 1`, `massless_neutrinos = 2.0307`, `nu_mass_degeneracies = 1.0153`.

Consider 2 massless active and 1 massive active neutrino all at the standard temperature $(3.046/3)^{1/4} T_\nu$, where the massive active neutrino has $m_i = 0.06\text{eV}$ and hence contributes $\Omega_\nu^a h^2 = 0.00064$. If there is also one sterile (at a different temperature) contributing ΔN_{eff} to the total so that $N_{\text{eff}} = 3.046 + \Delta N_{\text{eff}}$, and the total density parameter

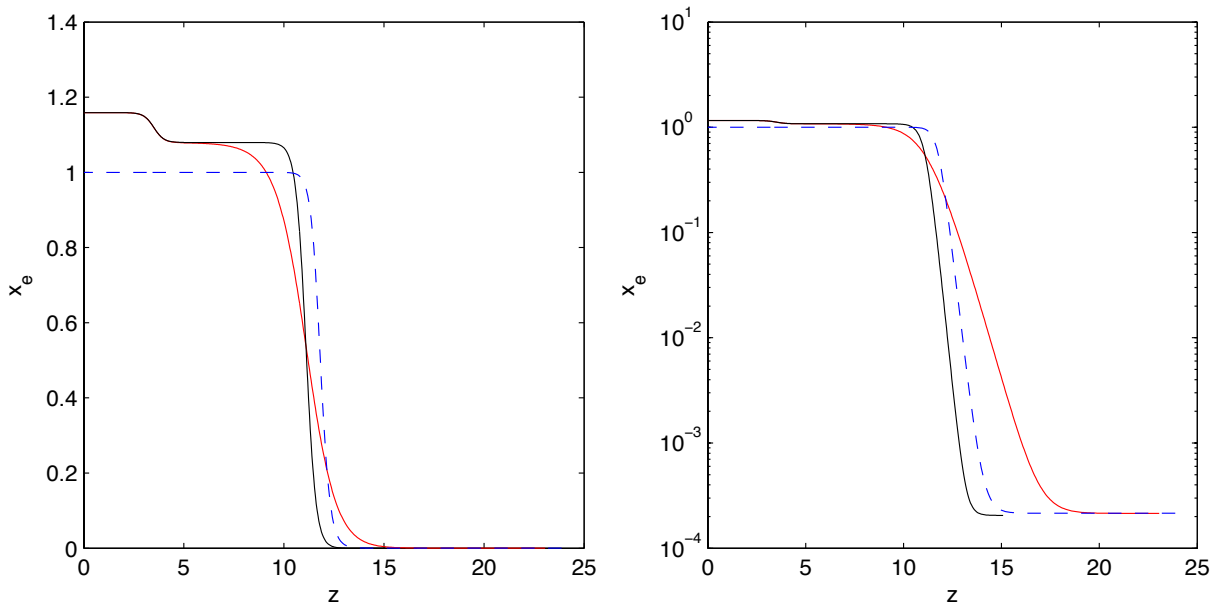


FIG. 1: Three recombination histories all with $\tau = 0.09$. The dashed line is the model typically used by CMBFAST and CAMB prior to March 2008 with $f = 1$. The black line is the new model with $\Delta_z = 0.5$, the red line with $\Delta_z = 1.5$.

$\Omega_\nu h^2 = 0.00064 + \Omega_\nu^s h^2$, then use: `share_delta_neff = F`, `massless_neutrinos = 2.0307`, `nu_mass_eigenstates = 2`, `massive_neutrinos = 1 1`, `nu_mass_degeneracies = 1.0153 \Delta N_{\text{eff}}`, `nu_mass_fractions = X Y` where $X = 0.00064/\Omega_\nu h^2$ and $Y = \Omega_\nu^s h^2/\Omega_\nu h^2$.

III. REIONIZATION

CAMB's reionization model is described in the appendix of [6] and reproduced here. The `reionization.f90` module can easily be changed for different models. The optical depth to reionization is defined by

$$\tau = \int_0^{\eta_0} d\eta S n_e^{\text{reion}} \sigma_T \quad (9)$$

where n_e^{reion} is the number density of free electrons produced by reionization and η_0 is the conformal time today. This is not quite the same as the number density of electrons at late times because there is a small residual ionization fraction from recombination. At the level of precision required we can neglect this difference ($\lesssim 10^{-3}$), though CAMB keeps the ionization history smooth by mapping smoothly onto the recombination-residual.

Since $n_e \propto (1+z)^3 x_e(z)$, and reionization is expected to happen during matter domination,

$$\tau \propto \int dz x_e \sqrt{1+z} \propto \int d[(1+z)^{3/2}] x_e. \quad (10)$$

It is therefore handy to parameterize x_e as a function of $y \equiv (1+z)^{3/2}$. As of March 2008 CAMB's default parameterization uses a tanh-based fitting function

$$x_e(y) = \frac{f}{2} \left[1 + \tanh \left(\frac{y(z_{re}) - y}{\Delta_y} \right) \right], \quad (11)$$

where $y(z_{re}) = (1+z_{re})^{3/2}$ is where $x_e = f/2$: i.e. z_{re} measures where the reionization fraction is half of its maximum. Since the fitting function is antisymmetric about the mid point (and assuming it does not extend out of matter domination)

$$\tau = \int_0^{\eta_0} d\eta S n_e^{\text{reion}} \sigma_T \approx \text{akthom} \times f \int_0^{z_{re}} dz \frac{d\eta}{dS}. \quad (12)$$

In other words, with this parameterization the optical depth agrees with that for an instantaneous reionization model at the same z_{re} for all (matter-dominated) values of Δ_y (CAMB's variable `akthom` is $n_p\sigma_T$ today). Except in early dark energy models this result is quite accurate for the expected range of z_{re} . In practice the input parameter is Δ_z and Δ_y is taken to be $1.5\sqrt{1+z_{re}}\Delta_z$.

To keep the implementation general (e.g for early dark energy models), the default reionization module actually maps τ into z_{re} by doing a binary search. This method is also generally applicable for other monotonic models, e.g. the module also will work fine using a window function in a different power of $(1+z)$ (i.e. `Rionization_zexp` $\neq 3/2$), though in this case there is in general a more complicated relation between the optical depth and that of a sharp reionization model. Changing the exponent allows flexibility in relatively how quickly reionization starts and ends, with values $0.5 - 2.5$ changing the EE power by a couple of percent at fixed τ .

If hydrogen fully ionizes $f = 1$. However the first ionization energy of helium is similar, and it is often assumed that helium first re-ionizes in roughly the same way. In this case $f = 1 + f_{He}$, where $f_{He} = n_{He}/n_H$ is easily calculated from the input helium mass fraction Y_{He} . This is CAMB's default value of f ; typically $f \sim 1.08$.

In addition at $z \sim 3.5$ helium probably gets doubly ionized. Due to the low redshift this only affects the optical depth by ~ 0.001 , but for completeness this is included using a fixed tanh-like fitting function (modifying the above result for τ appropriately). Some reionization histories are shown in Fig. 1.

IV. INITIAL POWER SPECTRA

A. Scalar parameterization

CAMB uses a standard running power law model for the primordial super-horizon power spectrum $\mathcal{P}_s(k)$ of curvature perturbations (and similarly for isocurvature modes), with

$$\ln \mathcal{P}_s(k) = \ln A_s + (n_s - 1) \ln(k/k_s) + \frac{n_{\text{run}}}{2} [\ln(k/k_s)]^2 + \frac{n_{\text{run,run}}}{6} [\ln(k/k_s)]^3.$$

The input parameters are pivot scale `pivot_scalar` (k_s), and `scalar_spectral_index` (n_s), plus optional running parameters and `scalar_nrun` (n_{run}) and `scalar_nrunrun` ($n_{\text{run,run}}$). The amplitude at the pivot scale is set by `scalar_power_amp` (A_s). Wavenumber (k) scales are in Mpc^{-1} . You can generate results for multiple initial power spectra at one go, so all the input parameter (except pivot scales) are followed by the index, e.g. `scalar_power_amp(1)` for the first set.

B. Tensor parameterizations

CAMB supports three parameterizations of the primordial tensor power spectrum, set in the `.ini` file by `tensor_parameterization` integer variable. These use the input pivot scale `pivot_tensor` (k_t), and `tensor_spectral_index` (n_t). You can also specify a tensor running, `tensor_nrun` ($n_{t,\text{run}}$) so that

$$\ln \mathcal{P}_t(k) = \ln A_t + n_t \ln(k/k_t) + \frac{n_{t,\text{run}}}{2} [\ln(k/k_t)]^2.$$

The tensor power amplitude A_t is set depending on the parameterization, either using `initial_ratio` (r), or `tensor_amp` (A_t) directly:

- `tensor_parameterization = 1`

$$A_t = r A_s$$

The tensor amplitude here is independent of n_s and the scalar pivot scale (convenient if generating tensors separately), but does depend on A_s (`scalar_amp`). The parameterization only gives $P_t(k_t) = r P_s(k_t)$ if the tensor and scalar pivots are the same. This option is the default, and was used by CAMB prior to April 2014.

- `tensor_parameterization = 2`

$$A_t = r \mathcal{P}_s(k_t)$$

This allows the use of r defined at any given tensor pivot scale, and $P_h(k_t) = r P_s(k_t)$. However for different tensor/scalar pivot scales the result depends on n_s and the scalar pivot.

- `tensor_parameterization = 3`

Here `tensor_amp` gives the amplitude A_t directly.

This directly parameterizes the tensors in terms of their amplitude, which removes dependence on the shape and amplitude of the scalar spectrum (and is what tensor BB observations are actually probing directly).

C. Tensors in a closed universe

The power spectrum $\mathcal{P}_t = \mathcal{P}_h$ defined by the transverse traceless part of the metric tensor h_{ab} so that

$$\langle h_{ab}h^{ab} \rangle = \sum_{\nu} \frac{\nu}{\nu^2 - 1} \frac{\nu^2 - 4}{\nu^2} \mathcal{P}_h(\nu)$$

where the $\nu^2 - 4$ factor is the mode sum $l = 2 \dots \nu - 1$ of $2l + 1$. Note that $\nu^2/(\nu^2 - 1)$ is no longer the measure over $\ln k$ for tensor modes. The CMB result is then

$$C_l = \frac{\pi}{32} \frac{(l+1)(l+2)}{2l(l-1)} \sum_{\nu} \frac{\nu}{\nu^2 - 1} \frac{\nu^2 - 3}{\nu^2} \mathcal{P}_h(\nu) [T_I^{(l)}(\nu)]^2.$$

Internally CAMB uses the H_k , the metric tensor variable, and the shear σ_k . The relation between H_k , E_k (the electric part of the Weyl tensor) and the shear σ_k is

$$H_k = -\frac{\nu^2 - 3}{\nu^2 - 1} (2E_k + \sigma'_k/k)$$

and $H'_k = -k\sigma_k$. Using the series solution the relation between the initial E_k and H_k is

$$E_k = -\frac{2R_{\nu} + 10}{4R_{\nu} + 15} \frac{\nu^2 - 1}{\nu^2 - 3} H_k$$

where R_{ν} is the ratio of the neutrino and total radiation densities.

V. MULTIPOLE EQUATIONS, HARMONIC EXPANSION AND C_l

Here we summarize the equations of the covariant approach.

The photon multipole evolution is governed by the geodesic equation and Thomson scattering, giving [7]

$$\begin{aligned} \dot{I}_{A_l} + \frac{4}{3} \Theta I_{A_l} + D^b I_{bA_l} - \frac{l}{2l+1} D_{\langle a} I_{A_{l-1} \rangle} + \frac{4}{3} I A_{a_1} \delta_{l1} - \frac{8}{15} I \sigma_{a_1 a_2} \delta_{l2} \\ = -n_e \sigma_T \left(I_{A_l} - I \delta_{l0} - \frac{4}{3} I v_{a_1} \delta_{l1} - \frac{2}{15} \zeta_{a_1 a_2} \delta_{l2} \right) \end{aligned} \quad (13)$$

where I_{A_l} is taken to be zero for $l < 0$ and

$$\zeta_{ab} \equiv \frac{3}{4} I_{ab} + \frac{9}{2} \mathcal{E}_{ab} \quad (14)$$

is a source from the anisotropic stress and E-polarization. The equation for the density perturbation $D_a I$ is obtained by taking the spatial derivative of the above equation for $l = 0$. The corresponding evolution equations for the polarization multipole tensors are [7]

$$\begin{aligned} \dot{\mathcal{E}}_{A_l} + \frac{4}{3} \Theta \mathcal{E}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2} D^b \mathcal{E}_{bA_l} - \frac{l}{2l+1} D_{\langle a_l} \mathcal{E}_{A_{l-1} \rangle} - \frac{2}{l+1} \text{curl} \mathcal{B}_{A_l} = -n_e \sigma_T (\mathcal{E}_{A_l} - \frac{2}{15} \zeta_{a_1 a_2} \delta_{l2}) \\ \dot{\mathcal{B}}_{A_l} + \frac{4}{3} \Theta \mathcal{B}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2} D^b \mathcal{B}_{bA_l} - \frac{l}{2l+1} D_{\langle a_l} \mathcal{B}_{A_{l-1} \rangle} + \frac{2}{l+1} \text{curl} \mathcal{E}_{A_l} = 0. \end{aligned} \quad (15)$$

For numerical solution we expand the covariant equations into scalar, vector and tensor harmonics. The resulting equations for the modes at each wavenumber can be studied easily and also integrated numerically.

A. Scalar, vector, tensor decomposition

It is useful to do a decomposition into m -type tensors, scalar ($m = 0$), vector ($m = 1$) and 2-tensor ($m = 2$) modes. They describe respectively density perturbations, vorticity and gravitational waves. In general a rank- ℓ PSTF tensor X_{A_l} can be written as a sum of m -type tensors

$$X_{A_l} = \sum_{m=0}^l X_{A_l}^{(m)}. \quad (16)$$

Each $X_{A_l}^{(m)}$ can be written in terms of $l - m$ derivatives of a transverse tensor

$$X_{A_l}^{(m)} = D_{\langle A_{l-m}} \Sigma_{A_m \rangle} \quad (17)$$

where $D_{A_l} \equiv D_{a_1} D_{a_2} \dots D_{a_l}$ and Σ_{A_m} is first order, PSTF and transverse $D^{a_m} \Sigma_{A_{m-1} a_m} = 0$. The ‘scalar’ component is $X^{(0)}$, the ‘vector’ component is $X_a^{(1)}$, etc. Since GR gives no sources for X_{A_m} with $m > 2$ usually only scalars, vectors and (2-)tensors are considered. At linear order they evolve independently.

B. Harmonic expansion

For numerical work we perform a harmonic expansion in terms of zero order eigenfunctions of the Laplacian $Q_{A_m}^m$,

$$D^2 Q_{A_m}^m = \frac{k^2}{S^2} Q_{A_m}^m, \quad (18)$$

where $Q_{A_m}^m$ is transverse on all its indices, $D^{a_m} Q_{A_{m-1} a_m}^m = 0$. So a scalar would be expanded in terms of Q^0 , vectors in terms of Q_a^1 , etc. We usually suppress the labelling of the different harmonics with the same eigenvalue, but when a function depends only on the eigenvalue we write the argument explicitly, e.g. $f(k)$.

For $m > 0$ there are eigenfunctions with positive and negative parity, which we can write explicitly as $Q_{A_m}^{m\pm}$ when required. Since

$$D^2(\text{curl } Q_{A_m}) = \text{curl}(D^2 Q_{A_m}) = \frac{k^2}{S^2} \text{curl } Q_{A_m} \quad (19)$$

they are related by the curl operation. Using the result

$$\text{curl } \text{curl } Q_{A_m}^m = \frac{k^2}{S^2} \left[1 + (m+1) \frac{K}{k^2} \right] Q_{A_m}^m \quad (20)$$

we can choose to normalize the \pm harmonics the same way so that

$$\text{curl } Q_{A_m}^{m\pm} = \frac{k}{S} \sqrt{1 + (m+1) \frac{K}{k^2}} Q_{A_m}^{m\mp}. \quad (21)$$

A rank- ℓ PSTF tensor of either parity may be constructed from $Q_{A_m}^{m\pm}$ as

$$Q_{A_l}^m \equiv \left(\frac{S}{k} \right)^{l-m} D_{\langle A_{l-m}} Q_{A_m \rangle}^m \quad (22)$$

and a $X_{A_l}^{(m)}$ component of X_{A_l} may be expanded in terms of these tensors. They satisfy

$$\begin{aligned} D^2 Q_{A_l}^m &= \frac{k^2}{S^2} \left(1 - [l(l+1) - m(m+1)] \frac{K}{k^2} \right) Q_{A_l}^m \\ D^{a_l} Q_{A_{l-1} a_l}^m &= \beta_l^m \frac{k}{S} \frac{(l^2 - m^2)}{l(2l-1)} Q_{A_{l-1}}^m \\ \text{curl } Q_{A_l}^{m\pm} &= \sqrt{\beta_0^m} \frac{m}{l} \frac{k}{S} Q_{A_l}^{m\mp} \end{aligned} \quad (23)$$

where $\beta_l^m \equiv 1 - \{l^2 - (m+1)\} K/k^2$ and $l \geq m$.

Dimensionless harmonic coefficients are defined by

$$\begin{aligned}
\sigma_{ab}^{(m)} &= \sum_{k,\pm} \frac{k}{S} \sigma^{(m)\pm} Q_{ab}^{m\pm} & H_{ab}^{(m)} &= \sum_{k,\pm} \frac{k^2}{S^2} H^{(m)\pm} Q_{ab}^{m\pm} \\
q_a^{(m)} &= \sum_{k,\pm} q^{(m)\pm} Q_a^{m\pm} & E_{ab}^{(m)} &= \sum_{k,\pm} \frac{k^2}{S^2} E^{(m)\pm} Q_{ab}^{m\pm} \\
\pi_{ab}^{(m)} &= \sum_{k,\pm} \Pi^{(m)\pm} Q_{ab}^{m\pm} & I_{A_l}^{(m)} &= \rho_\gamma \sum_{k,\pm} I_l^{(m)\pm} Q_{A_l}^{m\pm} \\
\Omega_a &= \sum_{k,\pm} \frac{k}{S} \Omega^\pm Q_a^{1\pm} & A_a^{(m)} &= \sum_{k,\pm} \frac{k}{S} A^{(m)\pm} Q_a^{m\pm} \\
(D_a X)^{(m)} &= \sum_{k,\pm} \frac{k}{S} (\delta X)^{(m)\pm} Q_a^{m\pm}
\end{aligned} \tag{24}$$

where the k dependence of the harmonic coefficients is suppressed. We also often suppress m and \pm indices for clarity. The other multipoles are expanded in analogy with I_{A_l} . The heat flux for each fluid component is given by $q_i = (\rho_i + p_i)v_i$, where v_i is the velocity, and the total heat flux is given by $q = \sum_i q_i$. In the frame in which $\Omega_a = 0$ gradients are purely scalar $(\delta \bar{X})^{(1)} = 0$.

C. Harmonic multipole equations

Expanded into harmonics, the photon multipole equations (13) become

$$\begin{aligned}
I_l' + \frac{k}{2l+1} \left[\beta_{l+1}^m \frac{(l+1)^2 - m^2}{l+1} I_{l+1} - l I_{l-1} \right] = \\
- S n_e \sigma_T \left(I_l - \delta_{l0} I_0 - \frac{4}{3} \delta_{l1} v - \frac{2}{15} \zeta \delta_{l2} \right) + \frac{8}{15} k \sigma \delta_{l2} - 4h' \delta_{l0} - \frac{4}{3} k A \delta_{l1}
\end{aligned} \tag{25}$$

where $l \geq m$, $I_0 = \delta \rho_\gamma / \rho_\gamma$, $I_l = 0$ for $l < m$, and m superscripts are implicit. The scalar source is $h' = (\delta S/S)'$. The equation for the neutrino multipoles (after neutrino decoupling) is the same but without the Thomson scattering terms (for massive neutrinos see Ref. [8]). The polarization multipole equations (15) become

$$\begin{aligned}
\mathcal{E}_l^{m\pm'} + k \left[\beta_{l+1}^m \frac{(l+3)(l-1)}{(l+1)^3} \frac{(l+1)^2 - m^2}{(2l+1)} \mathcal{E}_{l+1}^{m\pm} - \frac{l}{2l+1} \mathcal{E}_{l-1}^{m\pm} - \frac{2m}{l(l+1)} \sqrt{\beta_0^m} \mathcal{B}_l^{m\mp} \right] = -S n_e \sigma_T (\mathcal{E}_l^{m\pm} - \frac{2}{15} \zeta^{m\pm} \delta_{l2}) \\
\mathcal{B}_l^{m\pm'} + k \left[\beta_{l+1}^m \frac{(l+3)(l-1)}{(l+1)^3} \frac{(l+1)^2 - m^2}{(2l+1)} \mathcal{B}_{l+1}^{m\pm} - \frac{l}{2l+1} \mathcal{B}_{l-1}^{m\pm} + \frac{2m}{l(l+1)} \sqrt{\beta_0^m} \mathcal{E}_l^{m\mp} \right] = 0.
\end{aligned} \tag{26}$$

D. Integral solutions

Solutions to the Boltzmann hierarchies can be found in terms of line of sight integrals. The flat vector and tensor results are given in Ref. [9]. General scalar and tensor results are in [7] (though I_l in that paper differs by a curvature factor).

E. Power spectra

Using the harmonic expansion of I_{A_l} the contribution to the C_l from type- m tensors becomes

$$C_l^{TT(m)} = \frac{\pi}{4} \frac{(2l)!}{(-2)^l (l!)^2} \sum_{k,k',\pm} \langle I_{l,k}^\pm I_{l,k'}^\pm \rangle Q_{A_l k}^\pm Q_{k'}^{A_l \pm}. \tag{27}$$

The multipoles I_l can be related to some primordial variable $X_{A_m} = \sum_k (X^+ Q_{A_m}^{m+} + X^- Q_{A_m}^{m-})$ via a transfer function T_l^X defined by $I_l = T_l^X X$. Statistical isotropy and orthogonality of the harmonics implies that

$$\langle X_k^\pm X_{k'}^\pm \rangle = f_X(k) \delta_{kk'} \quad (28)$$

where $\sum_k \delta_{kk'} Y_k = Y_{k'}$ and $f_X(k)$ is some function of the eigenvalue k . The normalization of the $Q_{A_l}^m$ is given by

$$\int dV Q_{A_l}^m Q^{m A_l} = \int dV \left(\frac{-S}{k} \right)^{l-m} Q_{A_m}^m D^{A_l - m} Q_{A_l}^m = \alpha_l^m \frac{(-2)^{l-m} (l+m)! (l-m)!}{(2l)!} N \quad (29)$$

where we have integrated by parts repeatedly, then repeatedly applied the identity for the divergence (23). The normalization is $N \equiv \int dV Q^{A_m} Q_{A_m}$ and $\alpha_l^m \equiv \prod_{n=m+1}^l \beta_n^m$. By statistical isotropy $C_l = (1/V) \int dV C_l$ and hence

$$C_l^{TT(m)} = \frac{\pi}{4} \frac{(l+m)! (l-m)!}{(-2)^m (l!)^2} \sum_{k, \pm} \frac{N}{V} \alpha_l^m |T_l^X(k)|^2 f_X(k) \quad (30)$$

We choose to define a power spectrum $P_X(k)$ so that the real space isotropic variance is given by

$$\langle |X_{A_m} X^{A_m}| \rangle = \sum_{k, \pm} \frac{|N|}{V} f_X(k) \equiv \int d \ln k P_X(k) \quad (31)$$

so the CMB power spectrum becomes

$$C_l^{TT(m)} = \frac{\pi}{4} \frac{(l+m)! (l-m)!}{2^m (l!)^2} \int d \ln k P_X(k) \alpha_l^m |T_l^X(k)|^2. \quad (32)$$

For a non-flat universe $\int d \ln k$ is replaced by some other measure which should be specified when defining the power spectrum. In a closed universe the integral becomes a sum over the discrete modes. Note that we have not had to choose a specific representation of Q_{A_m} or \sum_k .

The polarization C_l are obtained similarly [7] and in general we have

$$C_l^{JK(m)} = \frac{\pi}{4} \left[\frac{(l+1)(l+2)}{l(l-1)} \right]^{p/2} \frac{(2l)!}{(-2)^l (l!)^2} \frac{\langle J_{A_l} K^{A_l} \rangle}{\rho_\gamma^2} \quad (33)$$

$$= \frac{\pi}{4} \left[\frac{(l+1)(l+2)}{l(l-1)} \right]^{p/2} \frac{(l+m)! (l-m)!}{2^m (l!)^2} \int d \ln k P_X(k) \alpha_l^m J_l^X K_l^X \quad (34)$$

where JK is TT ($p = 0$), EE or BB ($p = 2$) or TE ($p = 1$). We have assumed a parity symmetric ensemble, so $C_l^{TB} = C_l^{EB} = 0$.

For tensors we use H_T where $h_{ij} = \sum_{k, \pm} 2H_T Q_{ij}^2$ and h_{ij} is the transverse traceless part of the metric tensor. This introduces an additional factor of $1/4$ into the result for the C_l in terms of P_h and $T_l^{H_T}$.

The numerical factors in the hierarchy and C_l equations depend on the choice of normalization for the ℓ and k expansions. Neither $e_{\langle A_l \rangle}$ nor Q_{A_l} are normalized, so there are compensating numerical factors in the expression for the C_l . If desired one can do normalized expansions, corresponding to an ℓ - and k -dependent re-scaling of the I_l and other harmonic coefficients, giving expressions in more manifest agreement with Ref. [10].

VI. QUINTESSENCE

CAMB uses a fluid model by default (see below), but a quintessence module is available separately.

The background field equation for a scalar field ψ can be written

$$\frac{d}{dS} (S^2 \psi') + \frac{S^3 V_{,\psi}}{\mathcal{H}} = 0.$$

This allows $\psi(S)$ and $\psi'(S)$ to be evaluated, and hence the background evolution since \mathcal{H} relates the time and S derivatives. The density and pressure are

$$S^2 \rho_\psi = \frac{1}{2} \psi'^2 + S^2 V \quad S^2 p_\psi = \frac{1}{2} \psi'^2 - S^2 V$$

The linearized exact field equation is

$$\psi'' + 2\mathcal{H}\psi' + D^a D_a \psi = -S^2 V_{,\psi}.$$

where D_a is the spatial covariant derivative. Defining the first order perturbation

$$\mathcal{V}_a \equiv S D_a \psi$$

and fixing to the zero-acceleration frame (the CDM frame) this implies the following equation for the perturbations

$$\mathcal{V}_a'' + 2\mathcal{H}\mathcal{V}_a' + S\mathcal{Z}_a\psi' + S^2 D_a D^b \mathcal{V}_b = -S^2 \mathcal{V}_a V_{,\psi\psi}.$$

This corrects the equation in [11], where other definitions are given. Performing the harmonic expansion one gets the scalar equation

$$\mathcal{V}'' + 2\mathcal{H}\mathcal{V}' + k\psi'\mathcal{Z} + k^2\mathcal{V} = -S^2 \mathcal{V} V_{,\psi\psi}.$$

This is what tells you that you can't consistently have $\mathcal{V} = 0$ in an evolving background if there are other perturbations present ($\mathcal{Z} \neq 0$). The heat flux and density perturbation $\mathcal{X}_a^{(\psi)} = S D_a \rho_\psi$ give the scalar relations

$$S^2 q = k\psi'\mathcal{V} \quad S^2 \mathcal{X}_\psi = \psi'\mathcal{V}' + S^2 \mathcal{V} V_{,\psi}.$$

There is no contribution to the anisotropic stress. CMBFAST/CAMB propagate quantities like $\kappa S^2 \rho$, hence it is useful to work with $\phi \equiv \sqrt{\kappa} \psi$ and similarly for \mathcal{V} .

Constant w

A useful parameterization is $w \equiv p_\psi/\rho_\psi = \text{constant}$, which implies

$$\rho_\psi = \rho_0 S^{-3(1+w)} \quad V = \frac{1}{2} \psi^2 \frac{1-w}{1+w}.$$

This potential and ψ' are easily obtained in terms of S

$$S^2 V = \frac{1-w}{2} S^2 \rho_\psi \quad \psi'^2 = |1+w| S^2 \rho_\psi.$$

For $w < -1$ we take the action to have a negative kinetic term, so

$$\psi'' + 2\mathcal{H}\psi' + D^a D_a \psi \pm S^2 V_{,\psi} = 0$$

etc, where \pm is the sign of $1+w$. The derivatives needed are then

$$V_{,\psi} = \mp \frac{3(1-w)}{2} H \psi \quad S^2 V_{,\psi} = \mp \frac{3(1-w)}{2} \mathcal{H} \psi',$$

$$V_{,\psi\psi} = \mp \frac{3(1-w)}{2} \left[\dot{H} - \frac{3}{2} H^2 (1+w) \right] \quad S^2 V_{,\psi\psi} = \mp \frac{3(1-w)}{2} \left[\mathcal{H}' - \frac{1}{2} \mathcal{H}^2 (5+3w) \right].$$

In many cases constant w is actually a very good approximation as far as the CMB is concerned, with [12]

$$w_{\text{eff}} \equiv \frac{\int da \Omega_\psi(a) w(a)}{\int da \Omega_\psi(a)}.$$

In quintessence domination $\rho \rightarrow \rho_\psi$ and $V_{,\psi\psi} \rightarrow \pm 9(1-w^2)H^2/2$.

Fluid equations

Following [13] the default CAMB module actually uses the fluid equations. For varying w these are

$$\delta_i' + 3\mathcal{H}(\hat{c}_{s,i}^2 - w_i)(\delta_i + 3\mathcal{H}(1+w_i)v_i/k) + (1+w_i)kv_i + 3\mathcal{H}w_i'v_i/k = -3(1+w_i)h' \quad (35)$$

$$v_i' + \mathcal{H}(1 - 3\hat{c}_{s,i}^2)v_i + kA = k\hat{c}_{s,i}^2\delta_i/(1+w_i), \quad (36)$$

where hatted quantities are evaluated in the frame co-moving with the dark energy. These equations are implemented in the code with $w' = 0$, and are equivalent to using the above, with the additional possibility of using the rest-frame sound speed \hat{c} different from one for generalized dark energy models ($\hat{c} = 1$ for quintessence)

VII. TIGHT COUPLING

At early times the baryons and photons are tightly coupled; the opacity $\tau_c^{-1} \equiv S n_e \sigma_T$ is large. This means $\delta \equiv q_\gamma - 4/3 v_b \approx 0$, and propagating the full photon hierarchy is impossible due to numerical problems in the source term proportional to $\tau_c^{-1} \delta$. The solution is to do an expansion in τ_c which is valid for $\epsilon \equiv \max(k\tau_c, \mathcal{H}\tau_c) \ll 1$. To first order in ϵ , assuming $c_s^2 \propto 1/S$ and dropping tiny terms in c_s^2 ,

$$v'_b = \frac{1}{1+R} \left(k c_s^2 \Delta_b + \frac{k}{4} R (\Delta_\gamma - 2\beta_2 \pi_\gamma) - \mathcal{H} v_b \right) \quad (37)$$

$$+ \left(\frac{2\mathcal{H}}{1+R} + (\ln \tau_c^{-1})' \right) \frac{3R}{4(1+R)} \delta - \frac{R\tau_c}{4(1+R)^2} (4(\mathcal{H}' + \mathcal{H}^2)v_b + k [2\mathcal{H}\Delta_\gamma + \Delta'_\gamma - 4c_s^2 \Delta'_b]) \quad (38)$$

$$\delta = \frac{\tau_c}{3(1+R)} (k\Delta_\gamma - 4k c_s^2 \Delta_b + 4\mathcal{H} v_b) \quad (39)$$

$$\pi_\gamma = \frac{32}{45} k \tau_c (v_b + \sigma) \quad (40)$$

$$E_2 = \frac{\pi_\gamma}{4} \quad (41)$$

where $R \equiv 4\rho_\gamma/3\rho_b$. Note that here I use the sign for E_2 consistent with that in the code, which is related to the polarization tensor by a minus sign. To a good, *but not good enough*, approximation $n_e \propto 1/S^3$ over the region of interest, so $(\ln \tau_c^{-1})' \approx -2\mathcal{H}$. However using this result does lead, for certain k , to a region in time in which neither the tight coupling approximation is accurate nor the full equations are stable. This is probably exacerbated by the RECFAST reionization history which doesn't change so abruptly at recombination. Using a numerical value for $(\ln \tau_c^{-1})'$ avoids the problem. Including the values for π_γ and E_2 makes the switch from tight coupling more robust, and allows it to be pushed to later times thereby speeding up the evolution on small scales.

A few terms from a next-order tight coupling expansion [14] are used where they make a significant difference, though the gain in terms of delaying the switch-off time is modest.

VIII. INITIAL CONDITIONS

For adiabatic modes we choose the initial conditions are set by the amplitude of χ , the comoving curvature perturbation (conserved on super-Hubble scales for the adiabatic mode). In general this is defined via the perturbation in the 3-Ricci scalar, $\eta_a \equiv \frac{1}{2} S D_a \mathcal{R}^{(3)}$ with harmonic coefficient η , on co-moving hypersurfaces (total heat flux $q = 0$, denoted by a bar), so that

$$\bar{\eta} = 2\beta_2 \left[\Phi + \frac{2}{3} \Omega^{-1} \frac{\mathcal{H}^{-1} \Phi' - \Psi}{1+w} \right] \equiv -2\beta_2 \chi.$$

where (for closed models) $\beta_2 = (\nu^2 - 4)/(\nu^2 - 1)$. Here Φ and Ψ are defined as in [15, 16], Ψ is the Newtonian potential, and $\Phi = \Phi_H$ is the potential satisfying the Poisson equation $\beta_2 k^2 \Phi = \frac{1}{2} \kappa S^2 \bar{\mathcal{X}}$. In general $\Phi = -\Psi - S^2 \kappa \Pi / k^2$. In flat models $\beta_2 = \Omega = 1$ and

$$\chi = -\bar{\eta}/2 = - \left[\Phi + \frac{2}{3} \frac{\mathcal{H}^{-1} \Phi' - \Psi}{1+w} \right].$$

For isocurvature modes the initial conditions are set for the corresponding non-zero mode (see later section). We assume statistical isotropy, and the initial power spectrum for the primordial variable X ($= \chi$ for adiabatic) on super-Hubble scales is defined so that

$$\langle |X|^2 \rangle = \sum_\nu \frac{\nu}{\nu^2 + 1} P_X(\nu).$$

Here P_χ gives the power in the 3-Ricci curvature by

$$\langle |\tilde{D}_a \tilde{R}^{(3)}|^2 \rangle = \sum_\nu \frac{\nu}{\nu^2 - 1} \frac{16k^6}{S^6} \left(\frac{\nu^2 - 4}{\nu^2 - 1} \right)^2 P_\chi(\nu). \quad (42)$$

During inflation (no anisotropic stress, $\Pi = 0$)

$$\chi = \psi + \frac{\mathcal{H}}{\phi'_i} \delta\phi_i = \Psi + \frac{2}{3} \Omega^{-1} \frac{\mathcal{H}^{-1} \Psi' + \Psi}{1+w}.$$

where ϕ_i is the inflaton field, and at the end of inflation $\chi = 3/2\Psi$.

In terms of the Weyl tensor variable ϕ (defined in [11], differing by a sign from Φ in [17])

$$\Phi = -\phi - \frac{1}{2} \frac{S^2}{k^2} \kappa \Pi \quad \Psi = \phi - \frac{1}{2} \frac{S^2}{k^2} \kappa \Pi.$$

In the early radiation dominated era, assuming purely adiabatic perturbations,

$$\chi = \frac{1}{2} \frac{4R_\nu + 15}{R_\nu + 5} \phi = \frac{4R_\nu + 15}{10} \Psi$$

where $R_\nu = \rho_\nu/(\rho_\nu + \rho_\gamma)$ (≈ 0.4). CMBFAST used to set the initial condition $\Psi = -1$, where Ψ is the metric perturbation ψ as defined in [18] in the conformal Newtonian gauge. In CAMB we set $\chi = -1$, same as the new CMBFAST default, so that the CAMB transfer functions are related to those of the old CMBFAST (in flat models) by $(4R_\nu + 15)/10$.

IX. REGULAR INITIAL CONDITION SERIES SOLUTIONS

Following¹ [20] we define $\omega \equiv \Omega_m \mathcal{H}_0 / \sqrt{\Omega_R}$, where $\Omega_R = \Omega_\gamma + \Omega_\nu$. The Friedmann equation gives

$$S = \frac{\Omega_m H_0^2}{\omega^2} \left(\omega\tau + \frac{1}{4} \omega^2 \tau^2 - \frac{1}{6} K \omega \tau^3 - \frac{1}{48} K \omega^2 \tau^4 + \mathcal{O}(\tau^5) \right).$$

Taking the lowest order in the tight coupling approximation we have the drag term

$$S n_e \sigma_T (4/3 v_b - q_\gamma) = - \frac{\rho_b}{3\rho_b + 4\rho_\gamma} (k\Delta_\gamma + 4\mathcal{H}v_b).$$

Higher orders corrections in $1/(n_e \sigma_T)$ are neglected (and hence π_γ and higher moments are zero, $v_b = 3q_\gamma/4$), and also assume $c_s^2 = p_b = 0$. We define $R_\nu = \Omega_\nu/\Omega_R$, $R_\gamma = \Omega_\gamma/\Omega_R$, $R_b = \Omega_b/\Omega_m$, $R_c = \Omega_c/\Omega_m$. The adiabatic mode (in the CDM frame — remember χ is defined in terms of η in the comoving frame, $\bar{\eta} = \eta + \mathcal{O}(k^2\tau^2)$) is, for $\chi_0 = -1$,

$$\eta = 2\beta_2 \left(1 - \frac{(k\tau)^2}{12} \left[\beta_2 - \frac{10}{4R_\nu + 15} \right] \right) \quad (43)$$

$$\Delta_\gamma = \Delta_\nu = \frac{\beta_2}{3} (k\tau)^2 - \frac{\beta_2}{15} \omega k^2 \tau^3 \quad (44)$$

$$\Delta_c = \Delta_b = \frac{\beta_2}{4} (k\tau)^2 - \frac{\beta_2}{20} \omega k^2 \tau^3 \quad (45)$$

$$q_\gamma = \frac{4}{3} v_b = \frac{\beta_2 (k\tau)^3}{27} \quad (46)$$

$$q_\nu = \frac{\beta_2 (k\tau)^3}{27} \frac{4R_\nu + 23}{4R_\nu + 15} \quad (47)$$

$$\pi_\nu = -\frac{4}{3} \frac{(k\tau)^2}{4R_\nu + 15} - \frac{\omega k^2 \tau^3}{3} \frac{4R_\nu - 5}{(4R_\nu + 15)(2R_\nu + 15)} \quad (48)$$

$$G_3 = -\frac{4}{21} \frac{(k\tau)^3}{4R_\nu + 15} \quad (49)$$

¹ The only difference here is that we give results for additional variables, and define the isocurvature modes following [19].

Further quantities not needed in the code are

$$\chi = -1 + \frac{(k\tau)^2}{12} \left(\beta_2 + \frac{4R_\nu - 5}{4R_\nu + 15} \right) \quad (50)$$

$$\Psi = \frac{-10}{4R_\nu + 15} - \frac{25\omega\tau}{8} \frac{(8R_\nu - 3)}{(4R_\nu + 15)(2R_\nu + 15)} \quad (51)$$

$$\frac{\kappa S^2 \Pi}{2k^2} = \frac{-2R_\nu}{4R_\nu + 15} + \frac{35}{2} \frac{\omega\tau R_\nu}{(2R_\nu + 15)(4R_\nu + 15)} \quad (52)$$

$$\sigma = \frac{-5k\tau}{4R_\nu + 15} - \frac{15\omega k\tau^2}{8} \frac{4R_\nu - 5}{(4R_\nu + 15)(2R_\nu + 15)} \quad (53)$$

$$\mathcal{Z} = \frac{3h'}{k} = -\frac{\beta_2}{2} k\tau + \frac{3\beta_2}{20} \omega k\tau^2. \quad (54)$$

Note that η and hence χ are constant to linear order, whereas the potential is only constant at zeroth order due to the matter changing the background equation of state.

CDM isocurvature mode:

$$\eta = \frac{\beta_2}{3} R_c \omega\tau - \frac{\beta_2}{8} R_c (\omega\tau)^2 \quad (55)$$

$$\Delta_c = 1 - \frac{1}{2} R_c \omega\tau + \frac{3}{16} R_c (\omega\tau)^2 \quad (56)$$

$$\Delta_b = -\frac{1}{2} R_c \omega\tau + \frac{3}{16} R_c (\omega\tau)^2 \quad (57)$$

$$\Delta_\gamma = \Delta_\nu = -\frac{2}{3} R_c \omega\tau + \frac{1}{4} R_c (\omega\tau)^2 \quad (58)$$

$$q_\gamma = q_\nu + \mathcal{O}(\tau^3) = -\frac{1}{9} R_c \omega k\tau^2 \quad (59)$$

$$\pi_\nu = -\frac{1}{3} \frac{R_c \omega k^2 \tau^3}{2R_\nu + 15} \quad (60)$$

$$\Phi = \frac{1}{8} \frac{R_c (4R_\nu + 15) \omega\tau}{(2R_\nu + 15)} \quad (61)$$

$$\sigma = \frac{1}{24} \frac{R_c (4R_\nu - 15) \omega k\tau^2}{2R_\nu + 15} \quad (62)$$

There is a solution $R_c \Delta_c = -R_b \Delta_b$ with everything else zero, since in this case there is no density perturbation and hence the dynamics is that of the background. The baryon isocurvature mode is given by subtracting ($\Delta_c = 1, \Delta_b = -R_c/R_b$) from the above mode, and is equivalent as far as the CMB is concerned so that $C_l(\text{CDMiso}, \Delta_{c0} = 1) = R_c^2/R_b^2 C_l(\text{baryoniso}, \Delta_{b0} = 1)$. The CDM isocurvature mode has a particularly simple form using the gauge invariant variables $\Delta_i^{\eta=0}$ (the perturbation in the frame in which $\eta = 0$, see later), with $\Delta_c^{\eta=0} = 1, \Delta_{\gamma,b,\nu}^{\eta=0} = 0$ to the above order.

We choose the neutrino isocurvature modes to have zero initial perturbation to the 3-Ricci scalar — a fairly natural definition of ‘isocurvature’. This implies that the density perturbation in the comoving frame (proportional to Φ) is non-zero, which makes an alternative definition of isocurvature, in which the result is a sum of the mode below and

the adiabatic mode. Neutrino isocurvature density mode:

$$\Delta_\gamma = -\frac{R_\nu}{R_\gamma} + \frac{R_\nu}{6R_\gamma}(k\tau)^2 \quad (63)$$

$$\Delta_\nu = 1 - \frac{(k\tau)^2}{6} \quad (64)$$

$$\Delta_c = -\frac{\omega k^2 \tau^3}{80} \frac{R_\nu R_b}{R_\gamma} \quad (65)$$

$$\Delta_b = \frac{1}{8} \frac{R_\nu (k\tau)^2}{R_\gamma} \quad (66)$$

$$q_\gamma = -\frac{R_\nu}{3R_\gamma} k\tau + \frac{\omega k \tau^2}{4} \frac{R_\nu R_b}{R_\gamma^2} \quad (67)$$

$$q_\nu = \frac{k\tau}{3} - \frac{(k\tau)^3}{54} \left(1 + \frac{12\beta_2}{4R_\nu + 15}\right) \quad (68)$$

$$\pi_\nu = \frac{(k\tau)^2}{4R_\nu + 15} \quad (69)$$

$$\eta = \frac{\beta_2 (k\tau)^2}{3} \frac{R_\nu}{4R_\nu + 15} \quad (70)$$

$$\Phi = -\frac{R_\nu}{4R_\nu + 15} - \frac{\omega\tau}{4} \frac{R_\nu(2R_\nu - 15)}{(4R_\nu + 15)(2R_\nu + 15)} \quad (71)$$

The weird neutrino velocity isocurvature mode is

$$\Delta_\gamma = k\tau \frac{R_\nu}{R_\gamma} - \frac{3\omega k \tau^2}{16} \frac{R_b(2 + R_\gamma)}{R_\gamma^2} \quad (72)$$

$$\Delta_\nu = -k\tau - \frac{3\omega k \tau^2 R_b}{16R_\gamma} \quad (73)$$

$$\Delta_c = -\frac{9\omega k \tau^2}{64} \frac{R_\nu R_b}{R_\gamma} \quad (74)$$

$$\Delta_b = \frac{3R_\nu}{4R_\gamma} k\tau - \frac{9\omega k \tau^2}{64} \frac{R_b(2 + R_\gamma)}{R_\gamma^2} \quad (75)$$

$$q_\gamma = -\frac{R_\nu}{R_\gamma} + \frac{3R_\nu R_b}{4R_\gamma^2} \omega\tau + \frac{(k\tau)^2}{6} \frac{R_\nu}{R_\gamma} + \frac{3(\omega\tau)^2}{16} \frac{R_\nu R_b}{R_\gamma^3} (R_\gamma - 3R_b) \quad (76)$$

$$q_\nu = 1 - \frac{(k\tau)^2}{6} \left(1 + \frac{4\beta_2}{4R_\nu + 5}\right) \quad (77)$$

$$\pi_\nu = \frac{2k\tau}{(4R_\nu + 5)} + \omega k \tau^2 \frac{6}{(4R_\nu + 5)(4R_\nu + 15)} \quad (78)$$

$$G_3 = \frac{3}{7} \frac{(k\tau)^2}{4R_\nu + 5} \quad (79)$$

$$\eta = 2\beta_2 k\tau \frac{R_\nu}{4R_\nu + 5} + \omega k \tau^2 \frac{3\beta_2 R_\nu}{32} \left(\frac{R_b}{R_\gamma} - \frac{80}{(4R_\nu + 5)(4R_\nu + 15)}\right) \quad (80)$$

$$\Phi = \Psi + \mathcal{O}(1) = -\frac{3R_\nu}{4R_\nu + 5} (k\tau)^{-1} \quad (81)$$

$$\phi = \frac{45}{4} \frac{R_\nu \omega}{k(4R_\nu + 15)(4R_\nu + 5)} \quad (82)$$

$$\sigma = -3 \frac{R_\nu}{4R_\nu + 5} + 2\phi k\tau \quad (83)$$

Note that the variables Φ and Ψ are singular. However these are only natural variables in the zero shear frame (the curvature and acceleration respectively), whereas ϕ is frame invariant and is regular. The results above generalize those of [19] for non-flat models (note that the adiabatic anisotropic stress term in [19] is wrong, as is the neutrino octopole in the neutrino velocity mode).

The quintessence field ψ is strongly damped by the high expansion rate in the early universe, and thus rapidly obtains very small velocity. Choosing the origin of the field $\psi(0) = 0$ we have $\psi(\tau) = -H_0^2 \Omega_R V_{,\psi} \tau^4 / 20 + \mathcal{O}(\tau^5)$, $w_\psi = -1 + H_0^2 \Omega_R V_{,\psi}^2 \tau^4 / (25V)$, and the above modes are unchanged on adding a quintessence field. There is an additional quintessence isocurvature mode with

$$\mathcal{V} = \frac{V}{V_{,\psi}} \left(1 - \frac{(k\tau)^2}{6} + \frac{\omega k^2 \tau^3}{72} \right) \quad (84)$$

$$\Delta_\psi = 1 - \frac{(k\tau)^2}{10} \quad (85)$$

and the other perturbations zero to this order. Note that fixing $w = \text{const}$ is inconsistent with these assumptions. However in the early universe the evolution is independent of the potential, so using the above initial conditions with $w = \text{const}$ is potentially not a bad approximation for various classes of quintessence models which start with $w \sim -1$, then evolve to have an effective constant $w = w_{\text{eff}}$.

The regular vector mode solution is given in Ref. [21] and the mode with magnetic fields in Ref. [9].

Frame invariant series

The isocurvature modes take a particularly simple form in the frame of identically vanishing curvature, in other words using the frame invariant variables $\hat{\Delta}_i = \Delta_i + \frac{3}{2\beta_2}(1 + w_i)\eta$, e.g. CDM isocurvature:

$$\hat{\Delta}_c = 1 - \frac{1}{72} \frac{R_c(4R_\nu - 15)\omega k^2 \tau^3}{2R_\nu + 15} \quad (86)$$

$$\hat{\Delta}_\gamma = \hat{\Delta}_\nu = \frac{5}{6} \frac{R_c \omega k^2 \tau^3}{2(R_\nu + 15)} \quad (87)$$

$$\hat{\Delta}_b = \frac{5}{8} \frac{R_c \omega k^2 \tau^3}{2(R_\nu + 15)} \quad (88)$$

$$v_b + \sigma = -\frac{15}{8} \frac{R_c \omega k \tau^2}{2R_\nu + 15} \quad (89)$$

$$v_c + \sigma = \frac{1}{24} \frac{R_c(4R_\nu - 15)\omega k \tau^2}{2R_\nu + 15} \quad (90)$$

$$q_\nu + \frac{4}{3}\sigma = -\frac{5}{2} \frac{R_c \omega k \tau^2}{2R_\nu + 15} \quad (91)$$

$$\Psi = \frac{1}{8} \frac{R_c(4R_\nu - 15)\omega \tau}{2R_\nu + 15} \quad (92)$$

(equal signs at this order only), consistent with comments below about conservation of the density perturbations in the zero curvature frame. See also Ref. [22].

X. MATTER POWER SPECTRUM

σ_8 is defined by

$$\sigma_8^2 = \frac{\langle |\int dV_3 \Delta(x)|^2 \rangle}{|\int dV_3|^2}$$

where Δ is the fractional total matter density perturbation (frame irrelevant at late times), and the integral is over a $8h^{-1}\text{Mpc}$ sphere. In spherical polar harmonics we have

$$\Delta = \sum_{lm} \int d \ln k \Delta_{klm} Y_{lmj_l}(kr),$$

then, since $Y_{00} = 1/\sqrt{4\pi}$,

$$\begin{aligned} \frac{\int dV_3 \Delta(x)}{\int dV_3} &= \frac{3}{4\pi r^3} \int d \ln k \int_0^r dr 4\pi r^2 \Delta_{k00} \frac{j_0(kr)}{\sqrt{4\pi}} \\ &= \frac{3}{\sqrt{4\pi}} \frac{kr \cos kr - \sin kr}{k^3 r^3} \Delta_{k00} \end{aligned}$$

where $r = 8h^{-1}\text{Mpc}$. Now by assumed statistical isotropy

$$\langle \Delta^2(x) \rangle = \langle \Delta^2(0) \rangle = \frac{1}{4\pi} \int d \ln k \int d \ln k' \langle \Delta_{k00} \Delta_{k'00} \rangle = \int d \ln k \mathcal{P}_\Delta$$

It follows that

$$\langle \Delta_{klm} \Delta_{k'l'm'} \rangle = 4\pi k \delta_{ll'} \delta_{mm'} \delta(k - k') \mathcal{P}_\Delta(k)$$

and hence

$$\sigma_8^2 = \int d \ln k \left[3 \frac{kr \cos kr - \sin kr}{k^3 r^3} \right]^2 T_\Delta^2 \mathcal{P}_\chi$$

where T_Δ is the transfer function.

We define the matter power spectrum so that

$$\langle \Delta^2(x) \rangle = \frac{1}{(2\pi)^3} \int d^3 k P_k = \int d \ln k \left(\frac{k^3 P_k}{2\pi^2} \right)$$

and is usually expressed in units of $h^{-1}\text{Mpc}$. Therefore

$$P_k = \frac{2\pi^2 \mathcal{P}_\chi}{k^3} T_\Delta^2.$$

For COBE-normalized runs this can be compared with the output from CMBFAST using

$$P_k = 8\pi^3 h^3 k (TF)^2 \text{d2norm}$$

where TF is the quantity in the transfer function output file, and k is just k (the first column of the output file is k/h).

XI. SUB-HORIZON OSCILLATORY EVOLUTION

To compute the matter transfer functions at late time on small scales many oscillations of the photon and neutrino multipoles have to be integrated. The integration is in fact very inaccurate except on high accuracy settings because of the low- l truncation of the multipole equations. The effect after matter domination is rather small so this is a waste of time. However even though the oscillations are inaccurate, the mean values are non-zero (and about correct), so for accurate results it is important to capture the non-oscillatory behaviour of the velocities when the radiation density fraction is not entirely negligible.

Furthermore, the neutrinos only affect observables via the first few moments of their distribution, so we do not directly need $l > 2$. Integrating the hierarchy is therefore not required as long as the lowest multipoles can be calculated accurately enough. An approximate scheme can therefore save time even before matter domination, though in radiation domination the oscillations need to be tracked.

As from June 2011, CAMB adopts essentially the scheme of [23], which is a significant improvement on previous schemes that only applied in strict matter domination. I summarize the argument here in CAMB's notation, with (trivial) generalization to a non-flat universe.

The neutrinos have the synchronous-gauge scalar-mode Boltzmann hierarchy

$$J'_l = \frac{k}{2l+1} [lJ_{l-1} - \beta_{l+1}(l+1)J_{l+1}] + \frac{8}{15} k\sigma\delta_{l2} - \frac{4}{3} k\mathcal{Z}\delta_{l0},$$

where as before $h'_s = 6h' = 2k\mathcal{Z}$, i.e. $\mathcal{Z} = h'_s/2k$ where h_s is the synchronous gauge quantity. In CAMB's conventions $J_0 = \Delta_\nu$, $J_1 = q_\nu$, $J_2 = \pi_\nu$, and all variables are dimensionless. In the absence of sources the hierarchy is solved

by (ultra)spherical Bessel functions, Φ_l^q (in the conventions of Antony's thesis so they are indeed a solution to the hierarchy normalized this way). When $k\tau \gg 1$, there is a separation of scales between the sources for the hierarchy ($l \leq 2$), and $l \sim k\tau$. Ref [23] suggest using this to generate an approximate scheme for the evolution of the lowest multipoles when $k\tau \gg 1$. Mathematically we look for a series solution

$$J_l = A\Phi_l^q + \sum_i f_i^q(\tau)/(k\tau)^i.$$

(in general higher derivatives of Φ_l^q can also be added, e.g. for velocity sources). Denoting order in $1/(k\tau)$ with a superscript, this leads to solutions to first order

$$\Delta_\nu = A\Phi_0^q + \Delta^{(1)} + \dots$$

$$q_\nu = A\Phi_1^q - \frac{4}{3}\mathcal{Z} + \dots$$

$$\pi_\nu = A\Phi_2^q + \frac{1}{2\beta_2} \left(4\mathcal{Z}'/k + \Delta^{(1)} \right) + \dots$$

$$J_3 = A\Phi_3^q + \dots$$

(there is some choice in defining orders, but the result below is the same). Using the recursion relations for the Bessel functions this can be rearranged to give an equation for π_ν' :

$$\pi_\nu' = k \left(q_\nu + \frac{4}{3}\mathcal{Z} \right) - 3 \cot_K \pi_\nu = -\Delta_\nu' - 3 \cot_K \pi_\nu.$$

Here $\cot_K = 1/\tau$ in a flat model, otherwise the usual appropriate trigonometric generalization. The leading solution is independent of the unknown function $\Delta^{(1)}$ at this order, a property that does not hold at higher order. This equation can be used for high $k\tau$, requiring only three equations to be evolved; it can be more accurate than a non-approximate in the regime in which $k\tau > l_{\max}$ where reflections from l_{\max} give numerical errors in the full solution, though it is not accurate in dark energy domination. It is used for sub-horizon massless neutrinos until after recombination, in which case the oscillations don't need to be integrated at all as follows.

The density multipoles obey the exact equation

$$\Delta_\nu'' = \frac{k^2}{3} (2\beta_2\pi_\nu - \Delta_\nu) - \frac{4}{3}k\mathcal{Z}'.$$

If we neglect π_ν this gives the non-oscillatory part of the solution

$$\Delta_\nu \sim -4 \frac{\mathcal{Z}'}{k}.$$

Since $\mathcal{Z}' = -\mathcal{H}\mathcal{Z} - \kappa\delta\rho_{\text{tot}}/(2k)$ this is easy to compute without any evolution of neutrino multipoles. The solution for the heat flux is read off from before as $q_\nu \sim -4/3\mathcal{Z}$. A similar argument applies for the photons (when the multipoles are not needed for reionization scattering) giving

$$\Delta_\gamma \sim -4 \frac{\mathcal{Z}'}{k} - 4 \frac{S n_e \sigma_\tau}{k} (v_b + \mathcal{Z})$$

where the opacity correction ensures the mean evolution is tracked correctly during reionization. We take $q_\gamma = q_\nu$ as scattering corrections appear to be negligible.

XII. MASSIVE NEUTRINO EVOLUTION

This is described in the appendix of Ref. [24], and reproduced here. Massive neutrinos have scalar mode perturbations with comoving momentum q that evolve with

$$F_l' + \frac{kv}{2l+1} [(l+1)\beta_{l+1}F_{l+1} - lF_{l-1}] + \left[\delta_{2l} \frac{2}{15} k\sigma - \delta_{0l} h' \right] \frac{d \ln F}{d \ln q} = 0$$

where the time-dependent velocity is $v \equiv q/\epsilon$ and ϵ is the comoving energy. For convenience we evolve $\nu_l \equiv -4F_l/(\frac{d \ln F}{d \ln q})$, i.e.

$$\nu'_l = \frac{kv}{2l+1} [l\nu_{l-1} - \beta_{l+1}(l+1)\nu_{l+1}] + \frac{8}{15}k\sigma\delta_{l2} - \frac{4}{3}k\mathcal{Z}\delta_{l0}.$$

Conveniently when the neutrino is relativistic so $v = 1$ this is identical to the massless neutrino equation ($\nu_L = J_L$), and ν_l is independent of q : the only difference with massive neutrinos is that they travel at a different (time dependent) speed once the mass becomes important. The hierarchies are truncated at l_{\max} using

$$\nu'_l = kv\nu_{l-1} - (l+1)\cot_K \nu_l.$$

Evolution is started when neutrinos are highly relativistic. To get the leading correction we can write

$$\nu_l = J_l + \frac{m^2}{2q^2}\Delta J_l$$

and use the series expansion $v \approx 1 - a^2m^2/(2q^2)$ so that

$$\Delta J'_l = \frac{k}{2l+1} [l\Delta J_{l-1} - \beta_{l+1}(l+1)\Delta J_{l+1}] - \frac{ka^2}{2l+1} [lJ_{l-1} - \beta_{l+1}(l+1)J_{l+1}].$$

This lets us calculate the evolution of any q momentum mode while that mode has $|am/q| \ll 1$, which for light neutrinos and larger q can be a while: only one additional hierarchy has to be evolved (that for ΔJ_l) until modes start to become significantly non-relativistic, at which point the full mode equation must be integrated for a sample of momenta. The time saving from this approximation is not large in itself, but it does allow lower l_{\max} to be used when switching to integrating the momentum modes separately - without it the momentum modes need to be integrated from the beginning with roughly the same l_{\max} as the massless neutrinos. It also makes clear the leading q dependence of the perturbed distribution that we use below.

To choose the momenta for direct mode integration, we note that we need integrals of the form

$$\frac{1}{4} \int_0^\infty dq \frac{q^4 e^q}{(1+e^q)^2} v^w \nu_l$$

in order to calculate the density, heat flux, and other perturbations. In the perturbatively relativistic regime, so that we can do an expansion in $a^2m^2/(2q^2)$ as before, the integrals are sums of terms involving integrals of the form

$$\int_0^\infty dq \frac{q^4 e^q}{(1+e^q)^2} q^n.$$

At late times we also expect $v \sim q$, also giving terms roughly of this form, though the distribution has evolved away from anything simple. We do not attempt to integrate the distribution accurately in the intermediate sub-hubble regime where there can be oscillations in q : ignoring these seems to be harmless at required precision, presumably because averaged over time or k they are smoothed out. So the idea is to chose a sampling in q so that integrals with $n = -4, -2, 2$ are evaluated exactly, which gives a set of constraint equations for the points and weights that can be solved, and if more points are desired the solution can be made unique by adding other constraints or making a choice of a few points (e.g. at high q would expect Gauss-Laguerre point sampling to be nearly optimal [25]). For 3 points we find² the remarkably sparse sampling $q = (0.913201, 3.37517, 7.79184)$ produces results accurate at the 2×10^{-4} level with

$$\frac{1}{4} \int_0^\infty dq \frac{q^4 e^q}{(1+e^q)^2} v^w \nu_l \approx \sum_i K_i v^w \nu_l$$

and kernel weights $K = (0.0687359, 3.31435, 2.29911)$. A four-point sampling is accurate at the $< \times 10^{-4}$ level, e.g. with $q = (0.7, 2.62814, 5.90428, 12)$, $K = (0.0200251, 1.84539, 3.52736, 0.289427)$. For $k\tau \gg 1$ (as for massless neutrinos), and once significantly non-relativistic, l_{\max} can be reduced down to 2 or 3.

Once the neutrinos become very non-relativistic we can evolve velocity-integrated equations (i.e. a truncated fluid hierarchy); this is described in detail in Ref. [8] [specifically see the Appendix for the scalar modes].

² Mathematica: <http://camb.info/maple/NeutrinoIntegrationKernels.nb>

XIII. LENSING

The CMB lensing method is described in Ref. [26] (for a better and simpler derivation see [27]). The correlation integral is evaluated over a truncated range with apodization for speed. Tails of the unlensed spectra are filled in from a template where their contributions to the result are small, and hence parameter dependence is very weak. Note that you need to turn on non-linear lensing to get the large-scale lensed BB spectrum accurately, and also large `k_eta_max_scalar`.

Limber approximation and lensing

This can be useful for calculating the lensing potential power spectrum accurately on small scales (it is not used on large scales). For introduction to lensing and the Limber approximation in the notation here see [27]. The flat universe full linear theory result for the lensing power spectrum is

$$C_l^\psi = 4\pi \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) \left[\int_0^{\chi_*} d\chi S_\psi(k; \eta_0 - \chi) j_l(k\chi) \right]^2, \quad (93)$$

where the lensing source is given in terms of the transfer function for the Weyl potential by

$$S_\psi(k; \eta_0 - \chi) = 2T_\Psi(k; \eta_0 - \chi) \left(\frac{\chi_* - \chi}{\chi_* \chi} \right) \quad (94)$$

for $\chi < \chi_*$ and zero otherwise. Note that in matter domination the potentials are nearly constant, so $T_\Psi(k; \eta_0 - \chi)$ is nearly constant. In this approximation, and making the approximation that the integral goes to infinity, we can use

$$\int_0^\infty d\chi j_l(k\chi) \left(\frac{\chi_* - \chi}{\chi_* \chi} \right) = \frac{\sqrt{\pi}}{4} \left[\frac{\Gamma[l/2]}{\Gamma[(l+3)/2]} - \frac{2}{k\chi_*} \frac{\Gamma[(l+1)/2]}{\Gamma[(l+2)/2]} \right]. \quad (95)$$

(can also use χ_* limit, but result is a mess.) The potentials will of course change when dark energy becomes important, though the scale of variation may be large compared to the Bessel function frequency on small scales. The Limber approximation picks out $k \sim l/\chi$ and the Bessel functions vary much faster than the source on small scales. Using

$$\int k^2 dk j_l(k\chi) j_l(k\chi') = \frac{\pi}{2\chi^2} \delta(\chi - \chi'), \quad (96)$$

we can Limber-approximate C_l^ψ as

$$C_l^\psi \approx \frac{8\pi^2}{l^3} \int_0^{\chi_*} \chi d\chi \mathcal{P}_\Psi(l/\chi; \eta_0 - \chi) \left(\frac{\chi_* - \chi}{\chi_* \chi} \right)^2. \quad (97)$$

Rather than using this result directly it is convenient to write the integral in a form closer to the full result, approximating

$$\int_0^{\chi_*} d\chi S_\psi(k; \eta_0 - \chi) j_l(k\chi) \approx S_\psi(k; \eta_0 - \chi_s) \int_0^{\chi_*} d\chi j_l(k\chi) \quad (98)$$

where $\chi_s = l/k$. Using the result

$$\int_0^\infty d\chi j_l(k\chi) = \frac{\sqrt{\pi}}{2k} \frac{\Gamma[(l+1)/2]}{\Gamma[(l+2)/2]} \sim \sqrt{\frac{\pi}{2l}} \frac{1}{k} \left(1 + \mathcal{O}\left(\frac{1}{l}\right) \right) \quad (99)$$

we have the large l flat universe approximation

$$C_l^\psi \approx 4\pi \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) \left[\sqrt{\frac{\pi}{2l}} \frac{1}{k} S_\psi(k; \eta_0 - \chi_s) \right]^2, \quad (100)$$

consistent with the previous results. For the non-flat case we have

$$C_l^\psi \approx \frac{8\pi^2}{l^3} \int_0^{\chi_*} f_K(\chi) d\chi \mathcal{P}_\Psi(l/f_K(\chi); \eta_0 - \chi) \left(\frac{f_K(\chi_* - \chi)}{f_K(\chi_*) f_K(\chi)} \right)^2, \quad (101)$$

which we can compute similar to the flat case by using the small scale source term given approximately by

$$\sqrt{\frac{\pi}{2l}} \frac{1}{k} \frac{1}{(1 - Kl^2/k^2)^{1/4}} S_\psi(k; \eta_0 - \chi_s) \quad (102)$$

where $f_K(\chi_s) = l/k$.

Accuracy is improved to higher order in the Limber results by using $l \rightarrow l + 1/2$, following Ref. [28].

XIV. NEUTRINO DAMPING ON TENSORS

It's well known that the effect of neutrino anisotropic stress can be neglected when computing the large scale temperature anisotropy. This is because the dominant large scale contribution ($\ell < 150$) comes from the anisotropy generated by viewing an isotropic last scattering surface through gravitational waves along the line of sight. The gravitational waves source photon anisotropies via their derivative $H' \sim \sigma$. On super-horizon scales $H' = 0$ and there is no contribution. On sub-horizon scales the waves decay, therefore the main contribution to the large scale anisotropy comes from when the waves come inside the horizon and start to decay, but before they start oscillating at which point they become small. After last scattering the evolution is matter dominated, so the neutrinos have almost no contribution to the energy density and hence the effect of neutrino anisotropic stress is negligible. For this reason CMBFAST, and by default CAMB, neglect the neutrinos as they slow down the computation.

For polarization things are different, as in this case the linear contribution to the anisotropy comes predominantly from the photon anisotropic stress sourced by H' . The large scale signal due to reionization is however insensitive to neutrinos as this is during matter domination, thus the polarization signal with highest signal to noise is also unaffected. On smaller scales the main contribution comes from the photon anisotropic stress at last scattering, which is affected by H' , itself a function of the damping due to neutrinos.

Ref. [29] has analysed the effect of neutrinos semi-analytically. In particular he finds a general semi-analytical result for modes which enter the horizon deep in radiation domination. Since this is before last scattering, and once inside the horizon the modes decay, this regime corresponds to the small oscillatory signal at high $\ell \gtrsim 200$. This is of little consequence for observations, though the effect is a substantial 35% reduction in power.

The polarization power spectra peak at $\ell \sim 100$, corresponding to modes which have maximal velocity at the time of last scattering. Therefore the most interesting region (apart from reionization) is on intermediate scales. Ref. [29] also analyses the case of modes which enter the horizon after radiation domination, and claims a $\sim 10\%$ damping of the power by analysing the effect of neutrinos on H' at last scattering. This accounts for the damping of individual modes and the phase difference relative to the fixed time of recombination.

See Figure. 2 for CAMB's output. The large scale effect of $\sim 5\%$ on $\ell < 100$ is numerically smaller than that in Ref. [29], and for $\ell \gtrsim 100$ the neutrinos actually increase the power slightly. My understanding is this: A mode which (without neutrinos) reaches maximal velocity at some time shortly after last scattering may be slowed by the damping such that the velocity *at the time of last scattering* is *higher* than without the damping, even though the overall amplitude of the mode is lower. Thus for modes on larger scales than the peak $\ell < \sim 100$ there should be a damping effect, but for $\ell \gtrsim 100$ there can be an increase in power at that ℓ

(CAMB has always supported tensor neutrinos by a flag in equationsxxx.f90; as of Dec 2003 this can also be set from the parameter input file)

XV. NOTES

The following are not strictly related to CAMB, but useful maybe to understand the equations (and S_ψ the obvious non-flat generalization).

A. Different Variables

As in my thesis I use total matter variables, e.g. $q = \sum_i \rho_i q_i$, $\mathcal{X} = \sum_i \rho_i \Delta_i$. In the co-moving $q = 0$ (zero heat flux, denoted by a bar) frame

$$\bar{\mathcal{X}} = \mathcal{X} + 3\mathcal{H}q/k \quad \bar{\sigma} = \sigma + \frac{q}{\rho + p} \quad \beta_2 \bar{\sigma} = \bar{\mathcal{Z}}$$

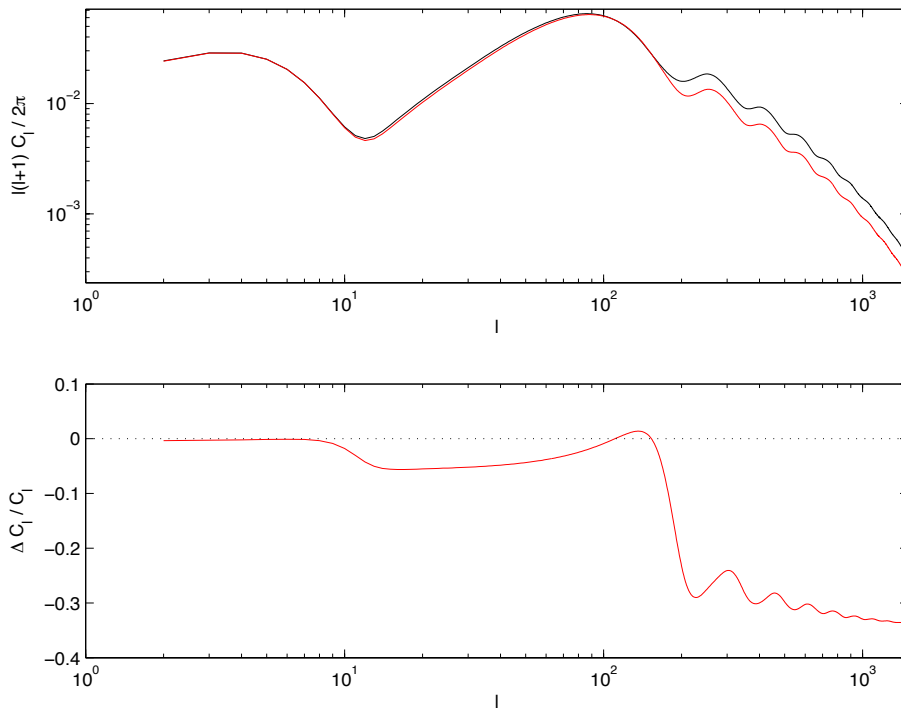


FIG. 2: The BB power spectrum with and without including the effect of neutrino damping (CAMB's `do_tensor_neutrinos` parameter)

are frame invariant, and hence

$$\bar{\eta} = \eta - \frac{2\beta_2 \mathcal{H} q}{k(\rho + p)}.$$

Using the frame invariance of $\bar{\mathcal{X}}$

$$\hat{q} = q + k\mathcal{X}/3\mathcal{H}$$

where a hat denotes the $\mathcal{X} = 0$ frame, and hence

$$\hat{\eta} = \eta + \frac{2\beta_2 \mathcal{X}}{3(\rho + p)} = \mathcal{H} \left(\frac{\eta}{\mathcal{H}} - \frac{2\beta_2 \mathcal{X}}{\dot{\rho}} \right)$$

is also frame invariant, so $\hat{\eta} = \eta_{\mathcal{X}=0} = \frac{2\beta_2}{3(\rho+p)} \mathcal{X}_{\eta=0}$. In terms of the flat metric variables as used by [30]

$$-\frac{1}{2}\hat{\eta} \sim -\zeta = \mathcal{H}\xi \equiv \mathcal{H} \left(\frac{\psi}{\mathcal{H}} + \frac{\delta\rho}{\dot{\rho}} \right)$$

and $\eta \sim -2\psi$. Since $\mathcal{X}/\dot{\rho}$ transforms like $\mathcal{X}_i/\dot{\rho}_i$ the curvature perturbation in the frame $\mathcal{X}_i = 0$ is the frame invariant quantity

$$\eta - 2\beta_2 \mathcal{H} \frac{\mathcal{X}_i}{\dot{\rho}_i}.$$

The energy conservation equation gives

$$3\hat{h}'_a(\rho + p) = -3\mathcal{H}\hat{\mathcal{X}}_a^p - S^2 \hat{D}_a \hat{D}^b \hat{q}_b.$$

which implies that \hat{h}_a is constant on large scales where the derivatives can be neglected if the pressure perturbation in the uniform energy density frame is zero, i.e. $\hat{\mathcal{X}}_a^p = (D_a p)_{\text{nad}} \equiv \mathcal{X}_a^p - p'/\rho' \mathcal{X}_a = 0$. As emphasised by [30] this follows purely from energy conservation, independently of the field equation (though in different theories the energy that is conserved may include components from the different theory). Using the general result

$$\eta' = -\frac{2\beta_2}{3}(k\sigma - 3h')$$

we have

$$\frac{1}{2\beta_2}\hat{\eta}' = -\mathcal{H}\frac{\delta p_{\text{nad}}}{\rho + p} - \frac{k\bar{\sigma}}{3}.$$

Hence on large scales $\hat{\eta}$ is conserved if $K = 0$ and $\delta p_{\text{nad}} = 0$. This applies equally well if the total densities are replaced with those for an individual species as long as they do not interact. For adiabatic modes $\hat{\eta} = \bar{\eta} + \bar{\mathcal{X}}\frac{2\beta_2}{3(\rho+p)} \approx \bar{\eta}$ on large scales since $\bar{\mathcal{X}}$ is suppressed by a factor of k^2 . The total and individual results are related by

$$\hat{\eta} = \frac{\sum_i(\rho_i + p_i)\hat{\eta}_i}{\sum_i(\rho_i + p_i)}.$$

Note that from the η' equation above, in the gauge in which $h' = 0$, η is conserved on large scales no matter what. Is this a useful statement, or like saying that in the $h' = 0$ gauge h is conserved!? Wands etc define this gauge by as the zero number density perturbation gauge, which amounts to the same thing as if $h' = 0$ the local volume element does not change with time and can be chosen to be zero initially. See also Ref. [22].

Newtonian Gauge

The Newtonian gauge is the frame with zero shear: $\sigma = 0$. To construct Newtonian gauge quantities from those in any other frame you just need to identify the gauge-invariant combinations that give what you want when the shear vanishes. See the Change of Frame section later for help constructing frame invariant combinations. In particular the Newtonian gauge velocity and density perturbations are given by

$$v_N^{(i)} = v^{(i)} + \sigma_i \quad \Delta_N^{(i)} = \Delta^{(i)} - 3\mathcal{H}(1 + w^{(i)})\sigma/k$$

where $w = p/\rho$. e.g. $\Delta_N^\gamma = \Delta^\gamma - 4\mathcal{H}\sigma/k$. In general

$$\Phi = \frac{\eta}{2\beta_2} + \frac{\mathcal{H}\sigma}{k}$$

and hence $2\beta_2\Phi = \eta_N$ is the curvature in the zero shear frame (Newtonian gauge). From the σ' equation $A_N = -\Psi$, and the η' relation $\eta'_N = 2\beta_2 h'_N$. A useful relation is

$$\left(\Delta_i + \frac{3(1 + w_i)}{2\beta_2}\eta\right)' = -k(1 + w_i)(v_i + \sigma).$$

See also the appendix of Ref. [22] and <http://cosmocoffee.info/viewtopic.php?t=212>.

B. Change of frame

In the frame $\tilde{u} = u + v$ we have

$$\begin{aligned} \tilde{q} &= q - (\rho + p)v \\ \tilde{\mathcal{X}} &= \mathcal{X} - \frac{\rho'v}{k} \\ \tilde{\eta} &= \eta - \frac{2\beta_2\mathcal{H}v}{k} \\ \tilde{\sigma} &= \sigma + v \\ \tilde{A} &= A + \frac{v' + \mathcal{H}v}{k} \\ \tilde{h}' &= h' + \frac{kv}{3} - \frac{(\mathcal{H}v)'}{k}. \end{aligned}$$

In terms of the covariant tensors we have (not checked for non-flat)

$$\begin{aligned}\tilde{\eta}_a &= \eta_a - 2\mathcal{H}D_aD^bv_b \\ \tilde{h}'_a &= h'_a + \frac{1}{3}S^2D_aD^bv_b - (\mathcal{H}v_a)' \\ \tilde{\delta}\rho_a &= \delta\rho_a - \rho'v_a \\ \tilde{\sigma}_{ab} &= \sigma_{ab} + D_{\langle a}v_{b\rangle}\end{aligned}$$

(note notation mess: $\delta\rho \equiv \mathcal{X}$) so e.g. a frame invariant curvature perturbation is

$$\eta_a - 2\mathcal{H}\frac{D_aD^b\delta\rho_b}{\rho'}. \quad (103)$$

C. Exact covariant GR reference

Using the $u_a u^a = 1$ signature.

Energy conservation:

$$\dot{\rho} + (\rho + p)\theta + D^a q_a - 2A^a q_a - \pi_{ab}\sigma^{ab} = 0$$

$$\dot{q}_{\langle a} + \frac{4}{3}\theta q_a + (\rho + p)A_a - D_a p + q^b \sigma_{ab} + q^b \varpi_{ba} + D^b \pi_{ab} - A^b \pi_{ab} = 0$$

Raychaudhuri equation:

$$\dot{\theta} + \frac{1}{3}\theta^2 + \frac{\kappa}{2}(\rho + 3p) - D^a A_a + \varpi^{ab}\varpi_{ba} + \sigma^{ab}\sigma_{ab} + A_a A^a = 0.$$

Gauss-Codazzi ($\varpi_{ab} = 0$ to define ${}^{(3)}\mathcal{R}$) implies:

$${}^{(3)}\mathcal{R} = 2\kappa\rho - \frac{2}{3}\theta^2 + \sigma_{ab}\sigma^{ab}.$$

Scale factor perturbation:

$$\dot{h}_a = \frac{1}{3}(Z_a - S\theta A_a) - (\sigma_{ab} + \varpi_{ab})h^b - 2u_a A^b h_b.$$

Define $\omega_a = \frac{1}{2}\text{curl } u_a$, so $\text{curl } \omega_a = D^b \varpi_{ba}$. Others:

$$D^b \varpi_{ba} - D^b \sigma_{ab} + \frac{2}{3}D_a \theta + \kappa q_a - 2A^b \varpi_{ba} = 0$$

$$D^a \omega_a + A^a \omega_a = 0$$

and the evolution equations

$$\dot{\sigma}_{\langle ab} + \frac{2}{3}\theta\sigma_{ab} - D_{\langle a}A_{b\rangle} + E_{ab} + \frac{1}{2}\kappa\pi_{ab} + \sigma_{\langle a}{}^c\sigma_{c\rangle b} + A_{\langle a}A_{b\rangle} + \omega_{\langle a}\omega_{b\rangle}$$

$$\dot{\omega}_{\langle a} + \frac{2}{3}\theta\omega_a = \frac{1}{2}\text{curl } A_a + \omega^b \sigma_{ab}$$

Here $\langle \rangle$ on the LHS is needed to project some time derivatives orthogonal to u_a .

D. Action (e.g. closed instanton)

We perform a harmonic expansion of a quantity X in the form

$$X = \sum_{klm} X_{klm} Q_{klm}$$

where $\Delta Q_{klm} = -k^2 Q^k$ and we choose $Q_{klm} = \Phi_l^\nu(\chi) Y_{lm}$ where we normalize the hyper-spherical Bessel functions so that $\Phi_0^\nu(0) = 1$ and $k^2 = K(\nu^2 - 1)$ (for closed models: everywhere $\nu^2 - 1 \rightarrow \nu^2 + 1$ for open models). For flat models $\Phi_l^\nu(r) = j_l(kr)$.

We expand a variable X in the form

$$X = \sum_k X_k Q^k$$

where k labels l, m and ν and $\Delta Q^k = -k^2 Q^k$. Mode expansion in the action gives

$$S = \frac{1}{2} \int dV_4 X \mathcal{L} X = \frac{N}{2} \int d\tau \sum_k X_k \mathcal{L} X_k$$

where

$$\int dV_3 Q^k Q^{k'} = N \delta_{kk'}$$

where $\delta_{kk'}$ is defined by $\sum_k \delta_{kk'} X_k = X_{k'}$. Spatial derivatives in \mathcal{L} are converted to ks . In a compact space the sum over modes is of the form

$$\sum_k = \sum_{\nu lm} A_\nu$$

where A_ν is some arbitrary ν -dependent factor. The expectation value of the mode coefficients is therefore given by

$$\langle |X_k(\tau)|^2 \rangle = \frac{1}{N A_\nu} G(\tau, \tau)$$

where the Greens' function is unambiguously determined by the real space operator and satisfies

$$\mathcal{L} G(\tau, \tau') = \delta(\tau - \tau').$$

The expectation value of each ν mode will be the same and

$$\langle |X(x, \tau)|^2 \rangle = \frac{\mathcal{K}^{3/2}}{2\pi^2} \sum_k G(\tau, \tau) / A_\nu = \sum_\nu \frac{\nu}{\nu^2 - 1} \mathcal{P}(\nu)$$

where

$$\mathcal{P}(\nu) \equiv \frac{\mathcal{K}^{3/2} \nu (\nu^2 - 1)}{2\pi^2} G(\tau, \tau)$$

is the power spectrum. The CMB power spectrum is given by

$$C_l = \frac{1}{16} \frac{1}{2l+1} \sum_k \frac{2N \mathcal{K}^{3/2} \nu^2}{\pi} \langle |X_k(\tau_r)|^2 \rangle [T_I^{(l)}(\nu)]^2$$

where $T_I^{(l)}(\nu)$ is the transfer function giving I_l when $X_k(\tau_r) = 1$ and r denotes the early radiation dominated era when the mode is still well outside the horizon. This gives

$$C_l = \frac{1}{16} \frac{2}{\pi} \sum_\nu \mathcal{K}^{3/2} \nu^2 G(\tau_r, \tau_r) [T_I^{(l)}(\nu)]^2 = \frac{4\pi}{16} \sum_\nu \frac{\nu}{\nu^2 - 1} \mathcal{P}_r(\nu) [T_I^{(l)}(\nu)]^2.$$

For the CMB Anthony uses $A_\nu = \mathcal{K}^{3/2}\nu^2$ and $N = \pi/2$ where $Q^k = \Phi_l^\nu(\chi)Y_{lm}$ and $\Phi_0^\nu(0) = 1$.

In the tensor case we have an action of the form

$$S = \frac{1}{8\kappa} \int dV_4 h^{ab} \mathcal{L} h_{ab}$$

and write

$$h_{ab} = \sum_k 2H_k Q_{ab}^k$$

where k labels ν, l, m and P , the parity. The power spectrum is given by

$$\mathcal{P}_h(\nu) \equiv \frac{4}{\pi^2} \mathcal{K}^{3/2} \nu (\nu^2 - 1) G(\tau, \tau)$$

where $\mathcal{L}G(\tau, \tau') = \delta(\tau - \tau')$. Note the additional factor of two difference from the additional sum over the two parities.

The contribution to the CMB is

$$C_l = \frac{1}{16} \frac{(l+1)(l+2)}{2l(l-1)} \frac{1}{2l+1} \sum_k \frac{2N\mathcal{K}^{3/2}\nu^2}{\pi} \frac{(\nu^2-3)}{\nu^2} \langle H_k^2 \rangle [T_l^{(l)}(\nu)]^2$$

where k labels positive parity modes only ($\bar{Q}_{Al}|_0 = 0$) giving the previous result.

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